On Hrushovski properties of Hrushovski constructions

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Definition (Extension property for partial automorphisms)

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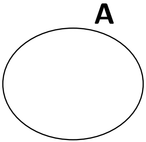

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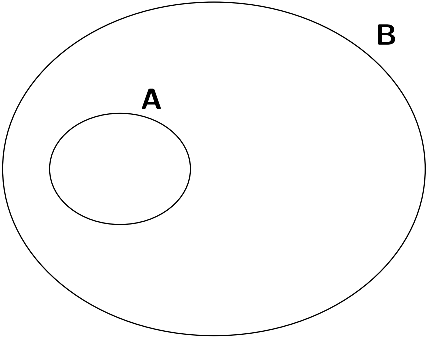

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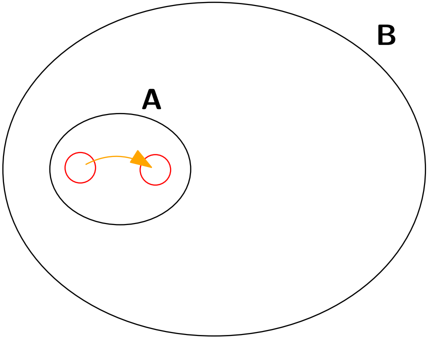

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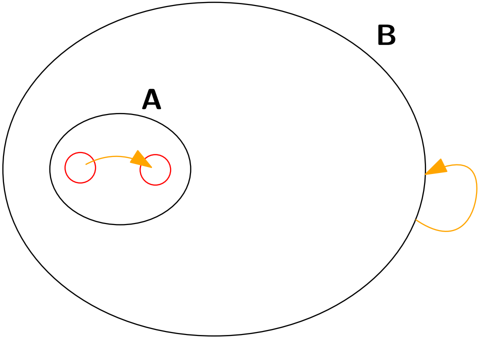

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Example (Classes with EPPA)

1. Graphs (Hrushovski 1992)
2. Relational structures (Herwig 1998)
3. Classes described by finite forbidden homomorphisms (Herwig-lascar 2000)
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   Ramsey with free linear order (Nešetřil-Rödl 1977, Abramson-Harrington 1978)
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- Predimension of a graph $\mathbf{G} = (V, E)$ is $\delta(G) = 2|V| - |E|$.

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$\delta(K_1) = 2$ $\quad$ $\delta(K_2) = 4 - 1 = 3$ $\quad$ $\delta(K_3) = 6 - 3 = 3$

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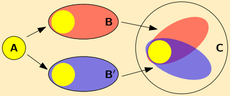

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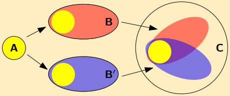
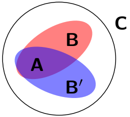


Lemma

$\mathcal{C}_0$ is closed for free amalgamation over self-sufficient substructures.



Proof.

$\delta(\mathbf{C}) = \delta(\mathbf{B}) + \delta(\mathbf{B}') - \delta(\mathbf{A})$. □

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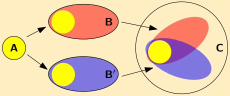

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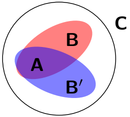

$\Longrightarrow$ $\mathcal{C}_0$ has a generalized Fraïssé limit $\mathbf{M}_0$.

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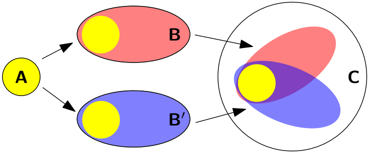
EPPA (with joint embedding) is a stronger form of amalgamation.



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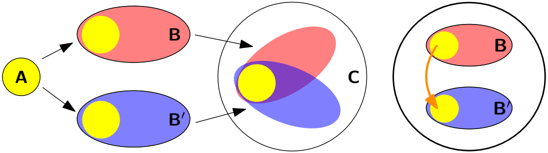

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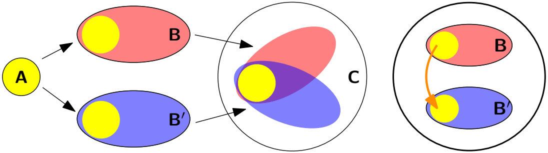

Question

Does class $\mathcal{C}_0$ have EPPA (or a Hrushovski property) for partial automorphisms of self-sufficient substructures?

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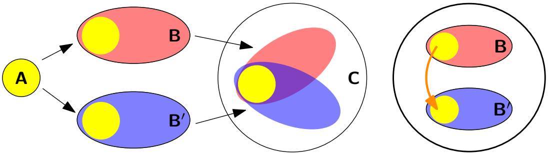

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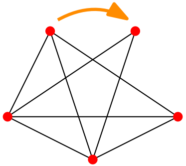

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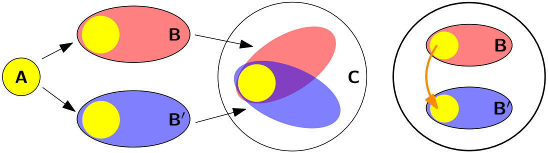

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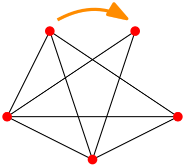

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In this talk we aim to understand the situation better.

Orientations $\mathcal{C}_0$

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- Finite graph $\mathbf{G}$ is in $\mathcal{C}_0$ iff $\forall_{\mathbf{H}\subseteq\mathbf{G}}\delta(\mathbf{H}) \geq 0$.

Lemma (By marriage theorem)

- $\mathbf{G} \in \mathcal{C}_0$ *iff it has 2-orientation (out-degrees at most 2).*
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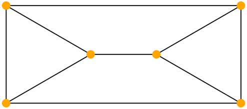

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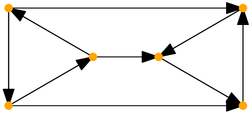

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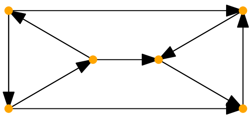

Corollary

$\mathcal{C}_0$ is, equivalently, created from class $\mathcal{D}_0$ of all finite 2-orientations by forgetting the orientation.

$\mathcal{D}_0$ is closed for free amalgamation over successor-closed substructures.

Hrushovski classes has no Hrushovski property

We use:

Theorem (Kechris, Rosendal 2007)

Suppose $\mathcal{K}$ is an amalgamation class of finite structures with (generalised) Fraïssé limit $\mathbf{M}$. Let $\Gamma = \mathrm{Aut}(\mathbf{M})$. Suppose $\mathcal{K}$ has EPPA then Γ is amenable.

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. . . and show:

Theorem (Evans, H., Nešetřil, 2019)

Let $\mathbf{M}_0$ be a generalised Fraïssé limit of $\mathcal{C}_0$. $\mathrm{Aut}(\mathbf{M}_0)$ is not amenable.

As a consequence of the two theorems $\mathcal{C}_0$ has no EPPA.

Non-amenability of $\mathbf{M}_0$

Theorem (Evans, H., Nešetřil, 2019)

Let $\mathbf{M}$ be an 2-orientable graph and Γ is a topological group which acts continuously on $\mathbf{M}$. Suppose there are adjacent vertices a, b in $\mathbf{M}$ such that the Γ_a-orbit containing b and the Γ_b-orbit containing a are both infinite. Then Γ is not amenable.

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Theorem (Evans, H., Nešetřil, 2019)

Let $\mathbf{M}$ be an 2-orientable graph and Γ is a topological group which acts continuously on $\mathbf{M}$. Suppose there are adjacent vertices a, b in $\mathbf{M}$ such that the Γ_a-orbit containing b and the Γ_b-orbit containing a are both infinite. Then Γ is not amenable.

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This is David Evans' argument generalised by Todor Tsankov.

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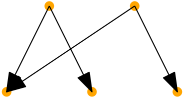

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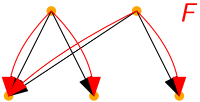

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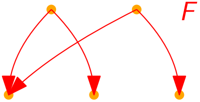

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Let L be a language consisting of relations and unary functions and $\mathcal{K}$ a free amalgamation class of L-structures. Then $\mathcal{K}$ has EPPA.

This is a strengthening of earlier result of Hodkinson and Otto for relational languages. Open for functions of arbitrary arity.

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What is maximal amenable subgroup of $\mathrm{Aut}(\mathbf{M}_0)$?

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- $F : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$

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Lemma

Put $F(x) = \ln(x)$. Then $\mathcal{C}_F$ is a free amalgamation class over d-closed substructures.

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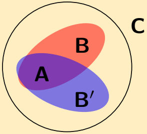

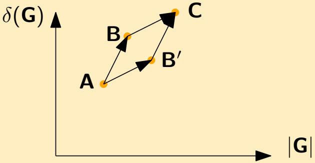

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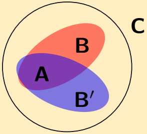

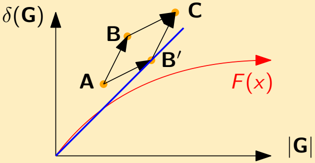

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Successor-d-closure

Denote by $\text{roots}_{\mathbf{A}}(\mathbf{B})$ the set of all roots of $\mathbf{A}$ reachable from $\mathbf{B} \subseteq \mathbf{A}$.

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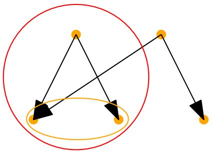

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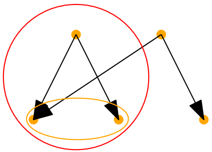

Proof.

- Given $\mathbf{B} \sqsubseteq_s \mathbf{A}$, $\delta(\mathbf{B})$ is the number of roots of out-degree 1 + twice number of roots of out-degree 0.
- Extending **B** by all vertices v such that $\text{roots}_{\mathbf{A}}(v) \subseteq \text{roots}_{\mathbf{A}}(\mathbf{B})$ does not affect δ.
- Extending **B** by any other vertex increases δ.

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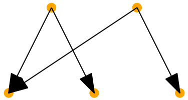

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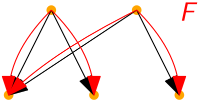

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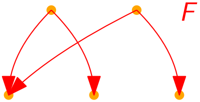

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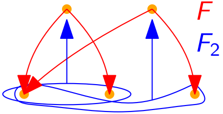

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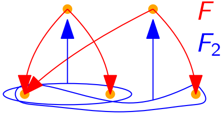

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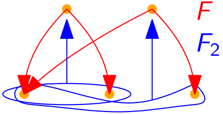

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Can we prove EPPA for a special case of free amalgamation class with non-unary functions?

Symmetric version of L-structures with partial functions and permutation of the language

- Let L be a language with relation symbols and function symbols each with arity denoted by $a(R)$ and $a(F)$.
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We consider Γ_L-structures which are essentially L-structures with the following definition of homomorphism:

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A homomorphism $f\colon \mathbf{A} \to \mathbf{B}$ is a pair $f = (f_L, f_A)$ where $f_L \in \Gamma_L$ and f_A is a mapping $A \to B$ such that for every $R \in L_{\mathcal{R}}$ and $F \in L_{\mathcal{F}}$ we have:

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(b) $f_A(F_{\mathbf{A}}(x_1, c_2, \ldots, x_{a(F)})) \subseteq f_L(F)_{\mathbf{B}}(f_A(x_1), f_A(x_2), \ldots, f_A(x_{a(F)}))$.

Notion of embedding, homomorphism-embedding, substructure generalise naturally to this category.

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Theorem (H., Konečný, Nešetřil 2019+)

Let Γ_L be a language equipped with a permutation group consisting of relations and unary functions and $\mathcal{K}$ a free amalgamation class of Γ_L-structures. Then $\mathcal{K}$ has EPPA. Moreover for every $\mathbf{A} \in \mathcal{K}$ the EPPA witness $\mathbf{B} \in \mathcal{K}$ can be constructed such that every irreducible substructure of $\mathbf{B}$ is isomorphic to a substructure of $\mathbf{A}$.

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6. Construct $\mathbf{B}'$ corresponding to $\mathbf{B}^+$ by adding the non-root vertices.

Summary and open problems

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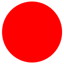
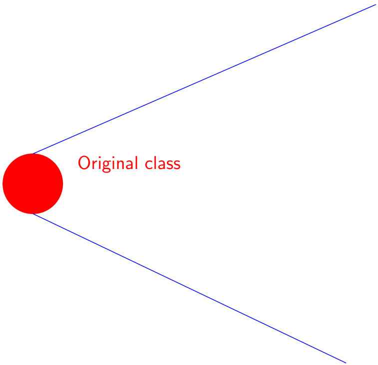
- Does the class of all finite structures with one quaternary relation defining two equivalence classes on pairs have EPPA?
- Can we obtain general structural condition for the existence of EPPA?



Original class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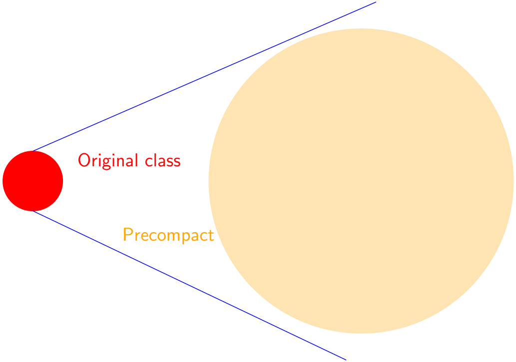
Original class
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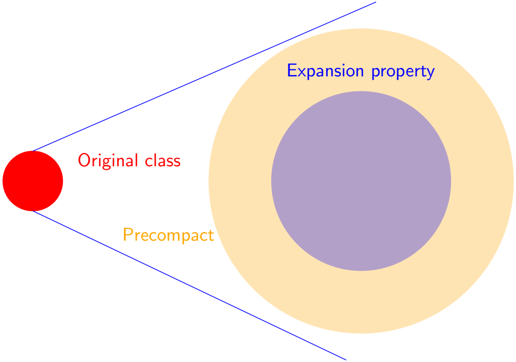
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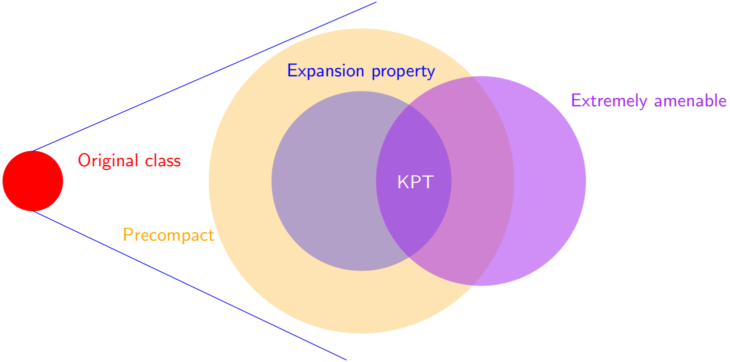
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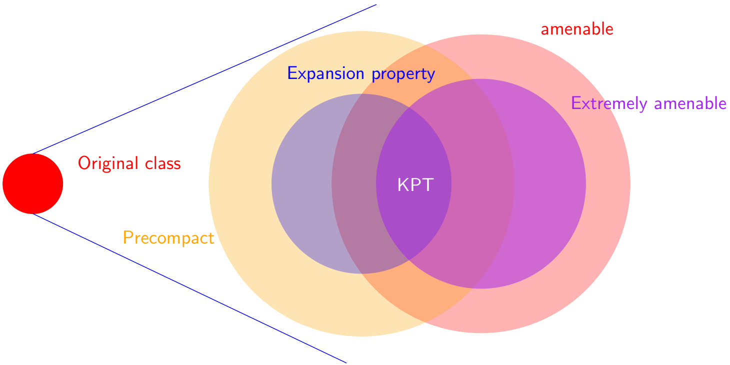
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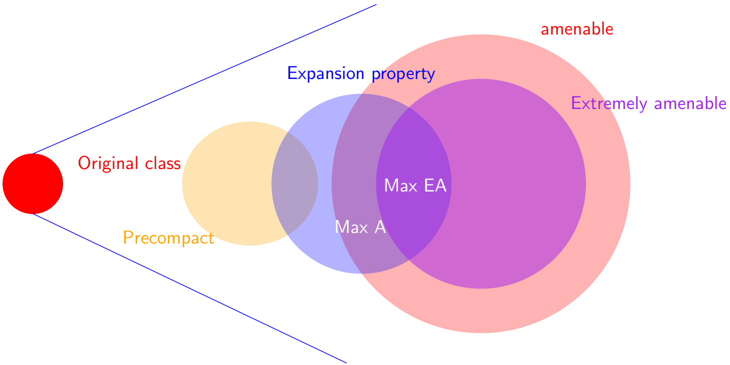
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Thank you for the attention

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