Assignment 1 (deadline: March 28)

The total number of points possible to obtain in this assignment: 68

Exercise 1. *Show that the limit permuton of every convergent sequence of permutations is unique.* [8 points]

Exercise 2. *Prove the Graph Removal Lemma for complete graphs K_k.* [5 points]

Exercise 3. *Prove the Graph Removal Lemma for all graphs.* [3 points] in addition the points for Exercise 2

Exercise 4. *Show that the Turán density of any graph H is equal to $\frac{\chi(H)-2}{\chi(H)-1}$.* [8 points]

Exercise 5. *Show that for every permuton μ the sequence of μ-random permutations with increasing sizes converges to μ with probability one.* [8 points]

Exercise 6. *Prove the following. For every $\varepsilon > 0$ and $k_0 \in \mathbb{N}$, there exists $K_0 \in \mathbb{N}$ such that for every graph G and every equipartition of $V(G)$ into at most k_0 parts, there exists an equipartition $V_1 \dot\cup \cdots \dot\cup V_k$ of $V(G)$ that refines the given equipartition of $V(G)$, has at most K_0 parts, i.e. $k \le K_0$, and all pairs of parts V_i and V_j, $1 \le i < j \le k$, except for at most εk^2 pairs are ε-regular, i.e. they satisfy that*

$$\left| \frac{e(A,B)}{|A|\ |B|} - \frac{e(V_i,V_j)}{|V_i|\ |V_J|} \right| \le \varepsilon$$

holds for all subsets $A \subseteq V_i$ and $B \subseteq V_j$ with $|A| \ge \varepsilon |V_i|$ and $|B| \ge \varepsilon |V_j|$. [6 points]

Exercise 7. *Prove the following. For every $\delta > 0$ and every graph H, there exists $\varepsilon > 0$ such that every ε-regular decomposition $V_1 \dot\cup \cdots \dot\cup V_k$ of any graph G satisfies that*

$$\left| t(H,G) - \frac{1}{k^{|V(H)|}} \sum_{f:V(H)\to[k]} \prod_{vw \in E(H)} d_{f(v)f(w)} \right| \le \delta$$

where $d_{ij} = \frac{e(V_i,V_j)}{|V_i|\ |V_j|}$ if $i \ne j$, and $d_{ii} = \frac{2\ |E(G[V_i])|}{|V_i|^2}$ if $i = j$. [8 points]

Exercise 8. *Show that if W is a graphon with $d(K_2,W) = 1/2$ and $d(K_3,W) = 0$, then there exists a partition of $[0,1]$ to two disjoint measurable sets A and B such that $W(x,y) = 0$ for almost every $(x,y) \in A^2 \cup B^2$ and $W(x,y) = 1$ for almost every $(x,y) \in [0,1]^2 \setminus (A^2 \cup B^2)$.* [6 points]

Exercise 9. *Show that any 2-edge-coloring of K_n contains at least $\frac{1}{4}\binom{n}{3} + o(n^3)$ monochromatic triangles.* [8 points]

Exercise 10. *Show that a sequence $(G_n)_{n\in\mathbb{N}}$ of tournaments is quasirandom if and only if $\lim_{n\to\infty} d(T_4,G_n) = \frac{4!}{2^6} = \frac{3}{8}$ where T_4 is the 4-vertex transitive tournament.* [8 points]