Lecture 11 (summary)

In this lecture, we give another example of the use of the flag algebra method. The example will relate to permutations and we will work without explicit mentioning the analytic representation of convergent sequences of permutations.

We will show the following theorem (the summary contains the main steps of the proof).

Theorem. *Every n-point permutation σ contains at $\frac{1}{4}\binom{n}{3} - o(n^3)$ monotone 3-point sub-permutations. i.e., $d(123, \sigma) + d(321, \sigma) \geq 1/4 - o(1)$.*

We use the notation for permutations as introduced earlier and, when we work with rooted permutations, the root will be underscored, e.g., $1\underline{3}2$ is the permutation 132 rooted at the point 3.

We start by computing the square of $(\underline{1}2 + 1\underline{2} - \underline{2}1 - \underline{2}1)^2$. The terms computing to this square are visualized in the next matrix (its entries are products of the labels of the corresponding row and the corresponding column).

	$\underline{1}2$	$1\underline{2}$	$-\underline{2}1$	$-2\underline{1}$
$\underline{1}2$	$\underline{1}23 + \underline{1}32$	$\frac{1}{2} \times 1\underline{2}3$	$-\frac{1}{2} \times \underline{2}13 - \frac{1}{2} \times \underline{2}31$	$-\frac{1}{2} \times 2\underline{1}3 - \frac{1}{2} \times 3\underline{1}2$
$1\underline{2}$	$\frac{1}{2} \times 1\underline{2}3$	$12\underline{3} + 21\underline{3}$	$-\frac{1}{2} \times 1\underline{3}2 - \frac{1}{2} \times 2\underline{3}1$	$-\frac{1}{2} \times 13\underline{2} - \frac{1}{2} \times 31\underline{2}$
$-\underline{2}1$	$-\frac{1}{2} \times \underline{2}13 - \frac{1}{2} \times \underline{2}31$	$-\frac{1}{2} \times 1\underline{3}2 - \frac{1}{2} \times 2\underline{3}1$	$\underline{3}12 + \underline{3}21$	$\frac{1}{2} \times 3\underline{2}1$
$-2\underline{1}$	$-\frac{1}{2} \times 2\underline{1}3 - \frac{1}{2} \times 3\underline{1}2$	$-\frac{1}{2} \times 13\underline{2} - \frac{1}{2} \times 31\underline{2}$	$\frac{1}{2} \times 3\underline{2}1$	$23\underline{1} + 32\underline{1}$

It follows that

$$[\![(\underline{1}2 + 1\underline{2} - \underline{2}1 - \underline{2}1)^2]\!] = \frac{1}{2} \times (123 + 321) - \frac{1}{6} \times (132 + 213 + 231 + 312) \geq 0,$$

i.e. $3 \times (123 + 321) - (132 + 213 + 231 + 312) \geq 0$. This inequality combines with $123 + 132 + 213 + 231 + 312 + 321 = 1$ to $4 \times (123 + 321) \geq 1$, which yields the desired inequality $123 + 321 \geq \frac{1}{4}$. Again, we use the notation that $a \geq \alpha$ where $a \in \mathcal{A}$ and $\alpha \in \mathbb{R}$ to denote that $\lim_{n\to\infty} d(a, \pi_n) \geq \alpha$ for every convergent sequences of permutations $(\pi_n)_{n\in\mathbb{N}}$ (or equivalently $d(a, \mu) \geq \alpha$ for every permuton μ).