Lecture 13 (summary)

In this lecture, we want to discuss the following theorem, which concerns a generalization of constant graphons to step graphons; we remark that for step graphons W, W-random graphs correspond to the stochastic block model studied in statistics.

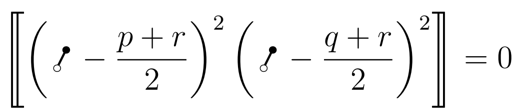
A graphon W is a step graphon if there exist measurable sets $A_1, \dots, A_k$ that partition the interval $[0,1]$ and reals $p_{i,j} \in [0,1]$, $1 \le i \le j \le k$, such that $W(x,y) = p_{i,j}$ for all $(x,y) \in A_i \times A_j$, $1 \le i \le j \le k$; we will usually write a_i for the measure of the set A_i, $i \in [k]$. For a particular choice of $\vec{a} \in [0,1]^k$ and $\vec{p} \in [0,1]^{\binom{k+1}{2}}$, the step graphon with these parameters such that $A_1, \dots, A_k$ are consecutive intervals of measures $a_1, \dots, a_k$ will be denoted by $W^{\vec{a},\vec{p}}$.

Similarly as constant graphons, step graphons are determined by finitely many densities of graphs.

Theorem (Lovász and Sós, 2008)**.** *For every $k \in \mathbb{N}$, $\vec{a} \in [0,1]^k$ and $\vec{p} \in [0,1]^{\binom{k+1}{2}}$ such that $a_1 + \cdots + a_k = 1$, there exists $N \in \mathbb{N}$ such that if a graphon W satisfies $d(H,W) = d(H, W^{\vec{a},\vec{p}})$ for all graphs H with at most N vertices, then there exists a partition of the interval $[0,1]$ to measurable sets $A_1, \dots, A_k$ such that the measure of A_i is a_i and $W(x,y) = p_{i,j}$ for almost every $(x,y) \in A_i \times A_j$, $1 \le i \le j \le k$.*

We sketch a proof of the theorem for $k = 2$ and $a_1 = a_2 = 1/2$; set $p = p_{11}$, $q = p_{22}$ and $r = p_{12}$. Throughout the proof, we will present various flag algebra identities that the graphon $W^{(1/2,1/2),(p,r,q)}$ satisfies and use them to derive the structure of a given (unknown) graphon W. The proof concludes by saying that any graphon satisfying these identities, in particular, any graphon W with the same density of every graph H appearing in these identities as $W^{(1/2,1/2),(p,r,q)}$, must have the structure that we have deduced.

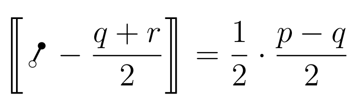
We first prove the theorem in the case when $p \ne q$. The first identity



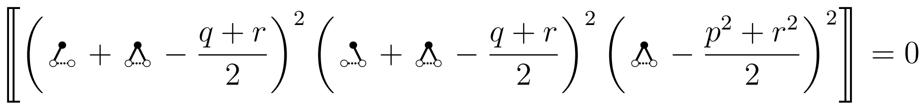
implies that

$$\int_{[0,1]} W(x,y)\,\mathrm{d}y \in \left\{ \frac{p+r}{2}, \frac{q+r}{2} \right\}$$

for almost every $x \in [0,1]$. Let A be the set of those x such that the above integral is equal to $\frac{p+r}{2}$ and $B = [0,1] \setminus A$. The second identity



yields that the measure of A is $1/2$. The third identity



implies that

$$\int_{[0,1]} W(x_1,y)W(x_2,y)\,\mathrm{d}y = \frac{p^2+r^2}{2}$$

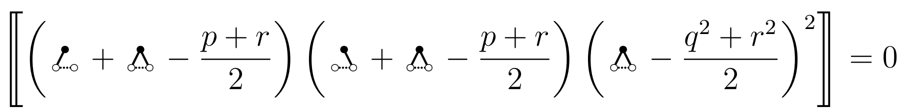
for almost every $(x_1,x_2)\in A^2$. This implies using the proposition from the previous lecture that

$$\int_{[0,1]} W(x,y)^2\,\mathrm{d}y = \frac{p^2+r^2}{2},$$

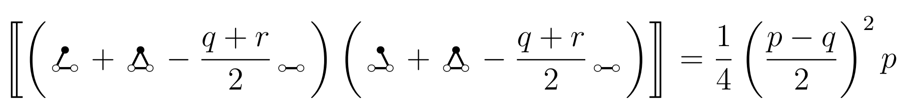
which yields that there exists a function $F_A:[0,1]\to[0,1]$ such that $W(x,y)=F_A(y)$ for almost every $(x,y)\in A\times[0,1]$. In particular, the set

$$\{(x_1,x_2,y)\in A^3, W(x_1,y)\neq W(x_2,y)\}$$

has measure zero, which implies that there exists p' such that $W(x,y)=p'$ for almost every $(x,y)\in A^2$. Along the same lines, we deduce from the identity

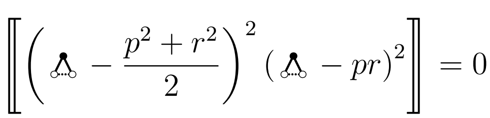


that there exists a function $F_B:[0,1]\to[0,1]$ such that $W(x,y)=F_B(y)$ for almost every $(x,y)\in B\times[0,1]$, which then yields that there exist q' and r' such that $W(x,y)=q'$ for almost every $(x,y)\in B^2$ and $W(x,y)=r'$ for almost every $(x,y)\in A\times B$. Note that $p+r=p'+r'$ and $q+r=q'+r'$. Finally, the identity



yields that $p'=p$ and so $q'=q$ and $r'=r$.

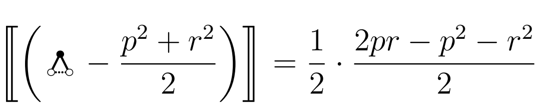
It remains to consider the case when $p=q$; we may assume $r\neq p$ (if $r=p$, we use the result from the previous lecture). The first identity



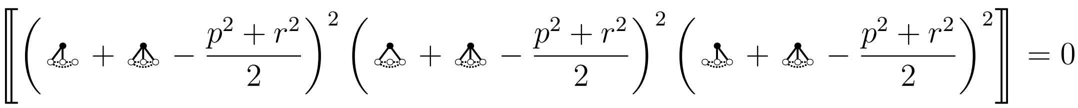
implies that

$$\int_{[0,1]} W(x_1,y)W(x_2,y)\,\mathrm{d}y \in \left\{\frac{p^2+r^2}{2}, pr\right\}$$

for almost every $(x_1,x_2)\in[0,1]^2$. Define an auxiliary graphon $W'(x_1,x_2)=1$ if the value of the above integral is $\frac{p^2+r^2}{2}$ and 0 otherwise. The identity



yields that $d(K_2, W') = 1/2$ and the identity



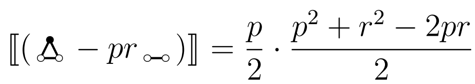
yield that $d(K_3, W') = 0$. It follows that there exist sets $A \subseteq [0,1]$ and $B = [0,1] \setminus A$ such that

$$\int_{[0,1]} W(x_1, y)W(x_2, y)\, \mathrm{d}y = \frac{p^2 + r^2}{2}$$

for almost all $(x_1, x_2) \in A^2 \cup B^2$ and

$$\int_{[0,1]} W(x_1, y)W(x_2, y)\, \mathrm{d}y = pr$$

for almost all $(x_1, x_2) \in A \times B$. It follows that there exist functions $F_A : [0,1] \to [0,1]$ and $F_B : [0,1] \to [0,1]$ such that $W(x,y) = F_A(y)$ for almost every $(x,y) \in A \times [0,1]$ and $W(x,y) = F_B(y)$ for almost every $(x,y) \in B \times [0,1]$, which implies that there exist p', q' and r' such that $W(x,y) = p'$ for almost every $(x,y) \in A^2$, $W(x,y) = q'$ for almost every $(x,y) \in B^2$ and $W(x,y) = r'$ for almost every $(x,y) \in A \times B$. It follows that $p^2 + r^2 = (p')^2 + (r')^2 = (q')^2 + (r')^2$ and $pr = p'r' = q'r'$, which implies that $p' = q'$ and $\{p, r\} = \{p', r'\}$. Finally, the identity



yields that $p = p'$ and so $r = r'$.

We remark that in the case $p \neq q$ we could use the "degrees" of $x \in [0,1]$ in the unknown graphon to identify which part of the graphon x belongs to; in the case $p = q$, this is not possible, however, once we have established the existence of the partition, we are able to identify whether two roots belong to the same part or to two different parts of the graphon.