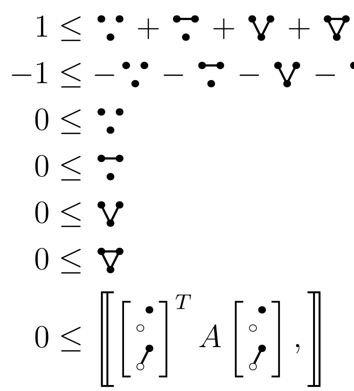
Lecture 14 (summary)

In this lecture, we show how to set up an SDP program to search for a flag algebra proof. We will demonstrate the approach on the following specific example: show that  +  $\geq \alpha$ for α as large as possible. We first fix a number of vertices $n \in \mathbb{N}$, which is the size of graphs that we work with. In our example, $n = 3$ (the choice of n corresponds to the computational power that we may use). List all graphs $G_1, \dots, G_K$ with that n vertices. We can combine the following three types of expression to get the desired inequality: $G_i \geq 0$, $G_1 + \dots + G_K = 1$, and $[\![v^T A v]\!]$ where A is a positive semidefinite matrix and v is a vector with entries from $\mathcal{A}^R$ for some root graph R, e.g, $R = \circ$ or $R = \circ\!\!-\!\!\circ$. As $G_1 + \dots + G_K = 1$ can be viewed as $G_1 + \dots + G_K \geq 1$ and $-G_1 - \dots - G_K \geq -1$, we are then searching for non-negative coefficients for the inequalities and suitable positive semidefinite matrices.

Recall that a semidefinite program is an optimization problem that asks to maximize $\langle C|X\rangle$ subject to $\langle A_1|X\rangle = b_1, \dots, \langle A_k|X\rangle = b_k$ over all positive semidefinite matrices X, where $\langle M|N\rangle$ for two matrices $M, N \in \mathbb{R}^{n,n}$ is the sum $M_{11}N_{11} + M_{12}N_{12} + \dots + M_{nn}N_{nn}$. Note that if X is positive semidefinite matrix, then every diagonal entry of X is non-negative and more generally every principal submatrix is positive semidefinite. So, we dedicate some of the diagonal entries of X to be the sought coefficients for the inequalities and some of the principal submatrices to be the sought positive semidefinite matrices. The constraints $\langle A_1|X\rangle = b_1, \dots, \langle A_k|X\rangle = b_k$ will then enforce the left hand side to as desired. Hence, we set $k = K$ and the i-th constraint will force the coefficient at G_i.

In our example, we have the following inequalities to combine:



where A is positive semidefinite. Let γ^+, γ^-, $\gamma_1, \dots, \gamma_4$ be the sought coefficients for the first inequality, we will think of the matrix X as

$$X = \begin{bmatrix} \gamma^+ & & & & & & & \\ & \gamma^- & & & & & & \\ & & \gamma_1 & & & & & \\ & & & \gamma_2 & & & & \\ & & & & \gamma_3 & & & \\ & & & & & \gamma_4 & & \\ & & & & & & A_{11} & A_{12} \\ & & & & & & A_{21} & A_{22} \end{bmatrix}.$$

Hence, the matrices $A_1, \ldots, A_4$, the reals $b_1, \ldots, b_4$ and the matrix C will set as follows.

$$A_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \text{and} \qquad b_1 = 1$$

$$A_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1/3 & 1/3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1/3 & 0 \end{bmatrix} \qquad \text{and} \qquad b_2 = 0$$

$$A_3 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1/3 & 1/3 \end{bmatrix} \qquad \text{and} \qquad b_3 = 0$$

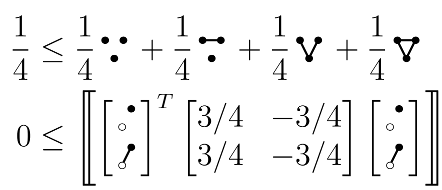
$$A_4 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad \text{and} \qquad b_4 = 1$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Running the program, we obtain that an optimal solution is the following matrix X:

$$X = \begin{bmatrix} 1/4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3/4 & -3/4 \\ 0 & 0 & 0 & 0 & 0 & 0 & -3/4 & 3/4 \end{bmatrix}$$

, which yields that the sum of the following two inequalities



gives the desired inequality,  +  $\geq 1/4$.