Lecture 16 (summary)

In this lecture, we are concerned with the following well-known problem, which is commonly known as Sidorenko's Conjecture and which was also made by Erdős and Simonovits.

Conjecture (Sidorenko's Conjecture)**.** *Every bipartite graph H satisfies that $t(H,W) \geq t(K_2,W)^{e(H)}$ for every graphon W.*

Those graphs H such that $t(H,W) \geq t(K_2,W)^{e(H)}$ holds for every graphon W are said to have the Sidorenko property. We will show that every complete bipartite graph $K_{a,b}$ has the Sidorenko property by proving the following statement.

Theorem. *Let $a, b \in \mathbb{N}$ and let G be a graph with n vertices and m edges. It holds that the number of homomorphisms from $K_{a,b}$ to G is at least*

$$\left(\frac{2m}{n^2}\right)^{ab} n^{a+b}.$$

We will use the entropy method, which was developed in this context by Szegedy. Recall that the entropy of a discrete random variable X is

$$H(X) = -\sum_x p(x) \log p(x)$$

where the sum ranges over all values x of X and $p(x)$ is the probability that $X = x$. If X takes one of n values, then $H(X) \leq \log n$. The conditional entropy of Y is the expected entropy of Y conditioned on the outcome of X, i.e.

$$H(Y|X) = -\sum_x p(x) \sum_y p(y|x) \log p(y|x).$$

It is straightforward to show that $H(X,Y) = X(Y|X) + H(X)$.

Fix $a, b \in \mathbb{N}$ and a graph G with n vertices and m edges. We will construct a suitable probability distribution on homomorphisms from $K_{a,b}$ to G. First, let E be the uniform distribution on *ordered* pairs of vertices (v, v') such that vv' is an edge of G; note that $H(E) = \log 2m$. Let X_1 be the probability distribution on vertices that is the projection of E to its first coordinate; note that the probability of a vertex in X_1 is proportional to its degree. Let $X_{1,a}$ be the probability distribution obtained as follows: choose v_0 according to X_1 and choose $v_1, \ldots, v_a$ uniformly among the neighbors of v_0 at random (independently of each other). Observe that (v_0, v_1) is a random edge of G and so $H(X_{1,a}) = a\, H(E|X_1) + H(X_1)$.

Let X_a be the probability distribution obtained from $X_{1,a}$ by restricting to $v_1, \ldots, v_a$. Finally, let $X_{a,b}$ be the probability distribution on homomorphisms from $K_{a,b}$ to G obtained as follows: choose an ordered a-tuple $v_1, \ldots, v_a$ of vertices according to X_a and then choose randomly vertices $w_1, \ldots, w_b$ among the common neighbors of $v_1, \ldots, v_a$ (independently of each other) with the same distribution as v_0 has in $X_{1,a}$ conditioned on the a-tuple being $v_1, \dots, v_a$.

We now estimate the entropy of the random variable $X_{a,b}$:

$$\begin{aligned} H(X_{a,b}) &= b\ H(X_{1,a}|X_a) + H(X_a) = bH(X_{1,a}) - (b-1)\ H(X_a) \\ &= b\ (a\ H(E|X_1) + H(X_1)) - (b-1)\ H(X_a) \\ &= ab\ H(E) - (a-1)b\ H(X_1) - (b-1)\ H(X_a) \\ &\geq ab\ H(E) - (a-1)b\log n - a(b-1)\log n \\ &= ab\log 2m - (a-1)b\log n - a(b-1)\log n = \log\left(\frac{2m}{n^2}\right)^{ab} n^{a+b}. \end{aligned}$$

It follows that the number of homomorphisms from $K_{a,b}$ to G is at least

$$\left(\frac{2m}{n^2}\right)^{ab} n^{a+b}.$$

The smallest graph for which the conjecture is open is the Möbius ladder on 10 vertices, i.e. the cubic (3-regular) graph obtained from $K_{5,5}$ by removing a Hamilton cycle.

It is possible to define a stronger property called the step Sidorenko property. We say that a graph H has the step Sidorenko property if every graphon W and every $k \in \mathbb{N}$ satisfies that $t(H, W) \geq t(H, W^{(k)}$ where the value of $W^{(k)}(x, y)$ for $x \in \left[\frac{i-1}{k}, \frac{i}{k}\right)$ and $y \in \left[\frac{j-1}{k}, \frac{j}{k}\right)$ is defined as

$$k^2 \int\limits_{\left[\frac{i-1}{k},\frac{i}{k}\right)\times\left[\frac{j-1}{k},\frac{j}{k}\right)} W(x,y)\ \mathrm{d}x\ \mathrm{d}y.$$

Clearly, if H has the step Sidorenko property, then H has the Sidorenko property (which is the above statement for $k = 1$). The step Sidorenko property is indeed stronger than the Sidorenko property, e.g. the 4×4 toroidal grid, i.e. the Cartesian product of C_4 and C_4, does not have the step Sidorenko property. On the other hand, it can be shown that every even cycle has the step Sidorenko property.

We say that a graph H is *weakly norming* if the following function is a norm on graphons, or equivalently on bounded symmetric function $U : [0, 1]^2 \to \mathbb{R}$:

$$\left(\int\limits_{[0,1]^{V(H)}} \left|\prod_{uv\in E(H)} U(x_u, x_v)\right| \mathrm{d}x_{V(H)}\right)^{1/e(H)}.$$

It can be shown that a graph H is weakly norming if and only if H has the step Sidorenko property.

Exercise. *Show that every tree has the Sidorenko property.*

Exercise. *Show that every even cycle has the Sidorenko property.*