Lecture 9 (summary)

In this lecture, we show that every convergent sequence of graphs has a limit graphon. To be able to do so, we will need two following statements, whose proofs are left as exercises. We say that a partition $V_1\dot\cup\cdots\dot\cup V_k$ is an *equipartition* if $|V_i - V_j| \le 1$ for all $1 \le i, j \le k$.

Exercise. *For every $\varepsilon > 0$ and $k_0 \in \mathbb{N}$, there exists $K_0 \in \mathbb{N}$ such that for every graph G and every equipartition of $V(G)$ into at most k_0 parts, there exists an equipartition $V_1\dot\cup\cdots\dot\cup V_k$ of $V(G)$ that refines the given equipartition of $V(G)$, has at most K_0 parts, i.e. $k \le K_0$, and all pairs of parts V_i and V_j, $1 \le i < j \le k$, except for at most εk^2 pairs are ε-regular, i.e. they satisfy that*

$$\left|\frac{e(A,B)}{|A|\ |B|} - \frac{e(V_i,V_j)}{|V_i|\ |V_J|}\right| \le \varepsilon$$

holds for all subsets $A \subseteq V_i$ and $B \subseteq V_j$ with $|A| \ge \varepsilon|V_i|$ and $|B| \ge \varepsilon|V_j|$.

We will refer to the decomposition (equipartition) of the vertex set of a graph G with the properties given in the first exercise as an *ε-regular decomposition* of G; for technical reasons, we will always assume that all parts of in an ε-regular decomposition are non-empty.

Exercise. *For every $\delta > 0$ and every graph H, there exists $\varepsilon > 0$ such that every ε-regular decomposition $V_1\dot\cup\cdots\dot\cup V_k$ of any graph G satisfies that*

$$\left|t(H,G) - \frac{1}{k^{|V(H)|}} \sum_{f:V(H)\to[k]} \prod_{vw\in E(H)} d_{f(v)f(w)}\right| \le \delta$$

where $d_{ij} = \frac{e(V_i,V_j)}{|V_i|\ |V_j|}$ if $i \ne j$, and $d_{ii} = \frac{2\,|E(G[V_i])|}{|V_i|^2}$ if $i = j$.

We next sketch the main steps of the proof of the following theorem; we follow the lines of a proof given by Lovász and Szegedy in 2006.

Theorem. *Every convergent sequence $(G_n)_{n\in\mathbb{N}}$ of graphs has a limit graphon.*

Sketch of proof. We first construct a sequence of mutually refining regular partitions in a suitably chosen subsequence of $(G_n)_{n\in\mathbb{N}}$. Let $\varepsilon_m = 2^{-m}$. We will describe the construction iteratively. To launch the iterative construction, we set $k^0 = 1$ and $V^0_{n,1} = V(G_n)$, and view $V^0_{n,1}$ as an equipartition of $V(G_n)$ to a single part.

Fix $m \in \mathbb{N}$. Apply the version of Szemerédi Regularity Lemma from the first exercise with ε_m and $k_0 = k^{m-1}/\varepsilon_m$ to get K_0. Split the equipartitions from the previous step to equipartitions with k^{m-1}/ε_m parts arbitrarily and apply Szemerédi Regularity Lemma for each graph G_n to obtain an ε_m-regular decomposition $V^m_{n,1}\dot\cup\cdots\dot\cup V^m_{n,k_n}$ of each graph G_n; we can assume that $V^{m-1}_{n,1}$ was split into the first k_n/k^{m-1} parts, $V^{m-1}_{n,2}$ to the next k_n/k^{m-1} parts, etc. By taking a subsequence, we may assume that k_n is equal to the same value for all sufficiently large n (note that $k_n \le K_0$); let k^m be this value.

Define a matrix $A_n^m \in [0,1]^{k^m \times k^m}$ to be the matrix such that its entry in the i-th row and the j-th column is equal to d_{ij} as defined in the second exercise with respect to the ε_m-regular decomposition $V_{n,1}^m \dot\cup \cdots \dot\cup V_{n,k^m}^m$ of G_n. By taking a subsequence, we assume that the matrices A_n^m converge entrywise; let A^m be the limit matrix. We now proceed to the next iteration (for $m+1$).

Based on A^m, we define a graphon W^m by splitting $[0,1]$ to k^m equal length intervals and setting the value of W^m on the product of the i-th interval and j-th interval to be A_{ij}^m. Note that the sequence $(W^m)_{m\in\mathbb{N}}$ can be viewed as a martingale on $[0,1]^2$. By Doob's First Martingale Convergence Theorem (or by Bounded Convergence Theorem), the sequence has a pointwise limit almost everywhere; let W be this limit function $[0,1]^2 \to [0,1]$, which we may assume to be symmetric. By Doob's Second Martingale Convergence Theorem, the sequence $(W^m)_{m\in\mathbb{N}}$ converges to W in $L^1[0,1]^2$.

Define $t(H,A)$ for a graph H and a matrix $A \in [0,1]^{s\times s}$ as

$$t(H,A) = \frac{1}{s^{|V(G)|}} \sum_{f:V(G)\to[s]} \prod_{vw\in E(G)} A_{f(v)f(w)}.$$

Next observe that $t(H,A^m) = t(H,W^m)$. The second exercise implies that for every $\delta > 0$, there exists m_0 and n_0 such that $|t(H,G_n) - t(H,A_n^m)| \le \delta$ for all $m \ge m_0$ and $n \ge n_0$, which implies that

$$|t(H,A^m) - \lim_{n\to\infty} t(H,G_n)| = |t(H,W^m) - \lim_{n\to\infty} t(H,G_n)| \le \delta$$

for every $m \ge m_0$. It follows that

$$\lim_{n\to\infty} t(H,G_n) = \lim_{m\to\infty} t(H,W^m).$$

Since W^m converge to W in $L^1[0,1]^2$, W is a limit graphon by the lemma stated at the end of this summary. □

Lemma. *Let W_1 and W_2 be two graphons and H a graph. It holds that $|t(H,W_1) - t(H,W_2)| \le |E(H)| \cdot \|W_1 - W_2\|_1$.*

The lemma can be proven along the following lines. Let $m = |E(H)|$, let $v_1w_1, \ldots, v_mw_m$ be the edges of H, and define τ_k for $k = 0, \ldots, |E(H)|$ as

$$\tau_k = \int_{[0,1]^{V(H)}} \prod_{i=1}^{k} W_1(x_{v_i}, x_{w_i}) \prod_{i=k+1}^{m} W_2(x_{v_i}, x_{w_i}) \, \mathrm{d}x_{V(H)}.$$

Observe that $\tau_0 = t(H,W_2)$, $\tau_m = t(H,W_1)$ and the following holds for every $k \in [m]$:

$$\begin{aligned}
|\tau_{k-1} - \tau_k| &= \left| \int_{[0,1]^{V(H)}} (W_2(x_{v_k}, x_{w_k}) - W_1(x_{v_k}, x_{w_k})) \prod_{i\ne k} W(x_{v_i}, x_{w_i}) \, \mathrm{d}x_{V(H)} \right| \\
&\le \int_{[0,1]^{V(H)}} |W_2(x_{v_k}, x_{w_k}) - W_1(x_{v_k}, x_{w_k})| \prod_{i\ne k} W(x_{v_i}, x_{w_i}) \, \mathrm{d}x_{V(H)} \\
&\le \int_{[0,1]^{V(H)}} |W_2(x_{v_k}, x_{w_k}) - W_1(x_{v_k}, x_{w_k})| \, \mathrm{d}x_{V(H)} = \|W_1 - W_2\|_1.
\end{aligned}$$