A New Quantum Lower-Bound Method, with Applications to Direct Product Theorems and Time-Space Tradeoffs

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joint work with Andris Ambainis† and Ronald de Wolf*

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Quantum algorithms

- Grover search: find a given number in an unsorted database of n records in time

$$O(\sqrt{n})$$

- element distinctness: find a collision $x_i = x_j$ in time

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Quantum query complexity

- allow quantum superposition, unitary evolution, and measurements
- count the number of queries, one query maps

$$|i, z\rangle \to (-1)^{x_i} |i, z\rangle$$

$i =$ queried bit

$z =$ the rest of memory

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- [Aaronson & Shi, 2002]
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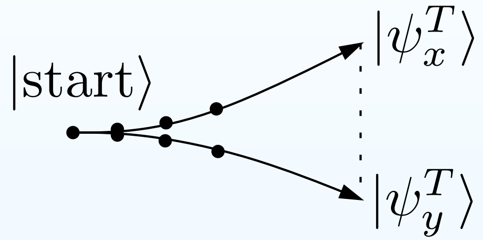

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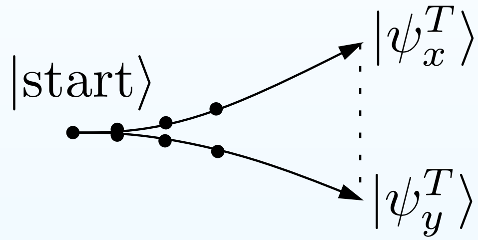

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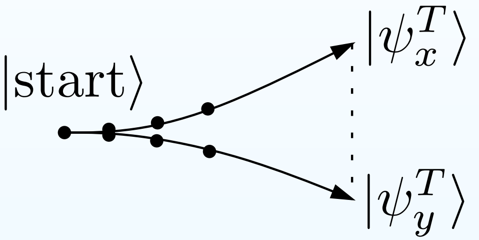

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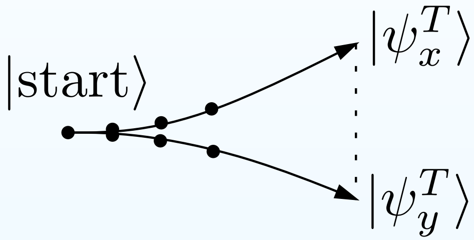

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2. [Š & Szegedy, Zhang, 2004]
   bounds limited by $\sqrt{C_0 C_1}$ for total functions
   C_z is the z-certificate complexity of f

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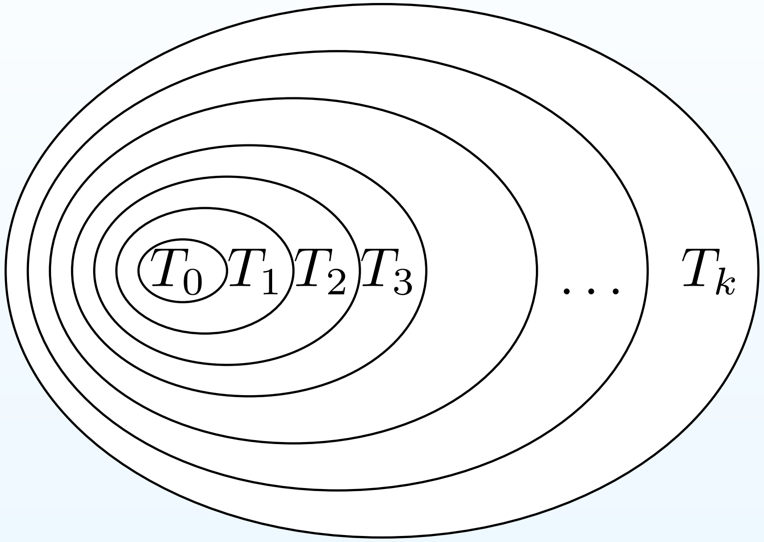
Applications

$$\left.\begin{array}{l} \bullet\ k\text{-fold search (find } k \text{ ones)} \\ \bullet\ \text{direct product theorems} \\ \bullet\ \text{time-space tradeoffs} \end{array}\right\} \text{explained in a moment}$$

Subspaces for k-fold search



$$T_0 \subseteq T_1 \subseteq \cdots \subseteq T_k$$

T_j "know" at most j ones

spanned by

$$|\psi_{i_1 \dots i_j}\rangle = \sum_{\substack{x:|x|=k \\ x_{i_1}=\cdots=x_{i_j}=1}} |x\rangle$$

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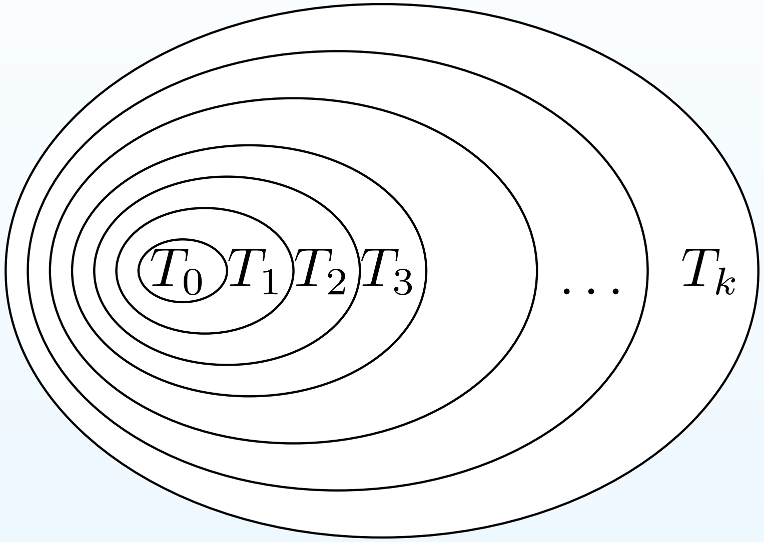

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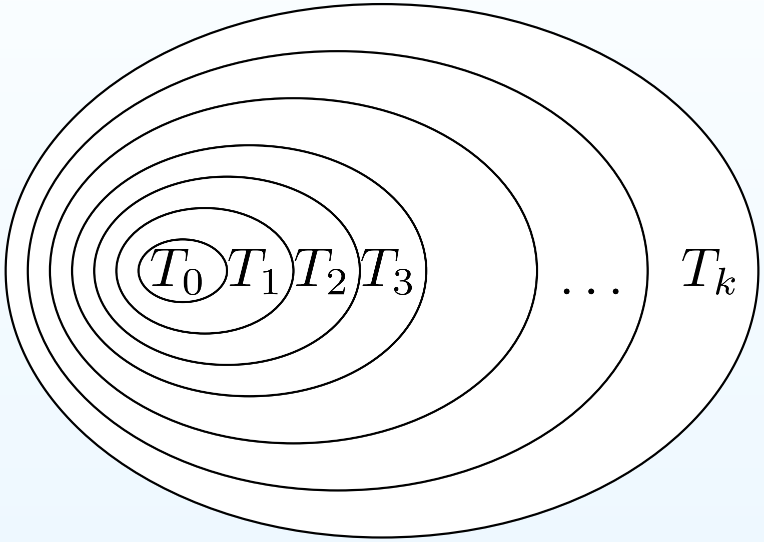

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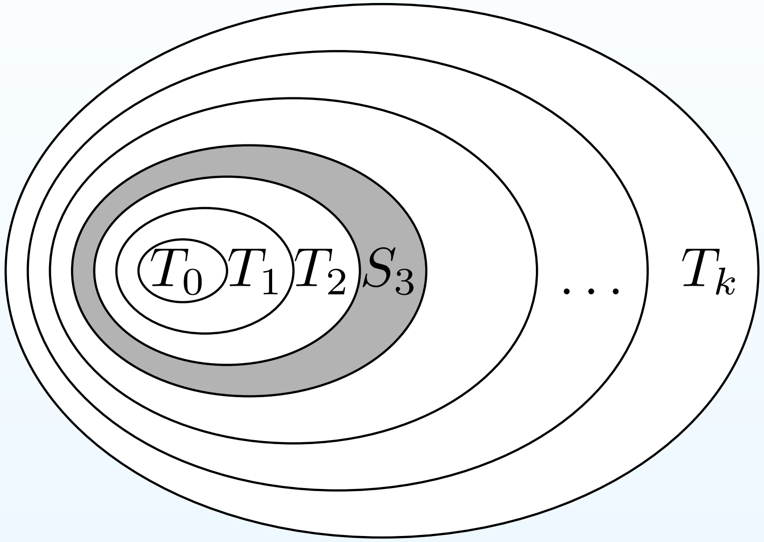

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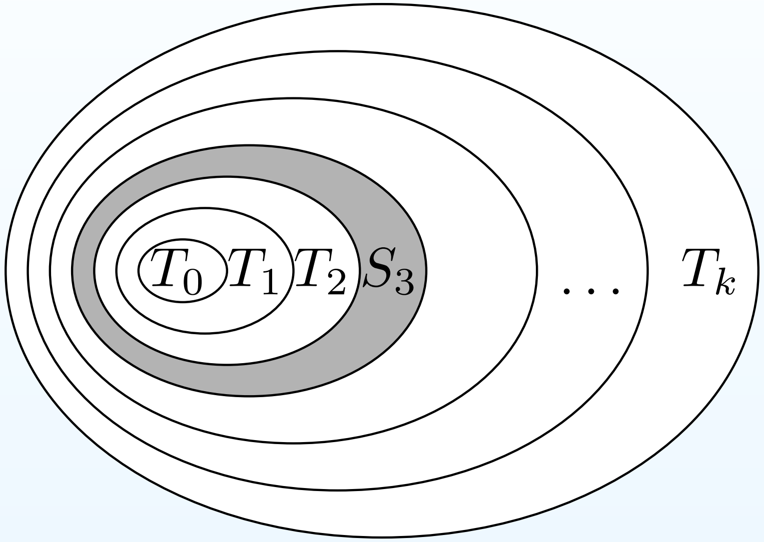

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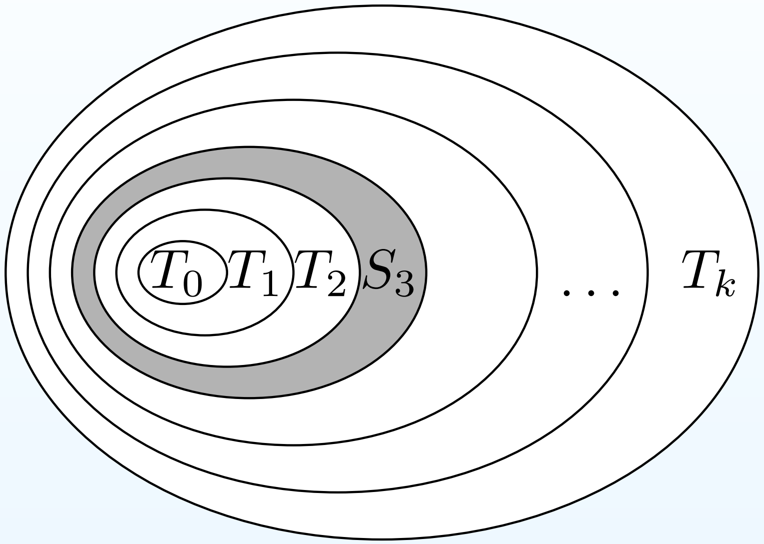

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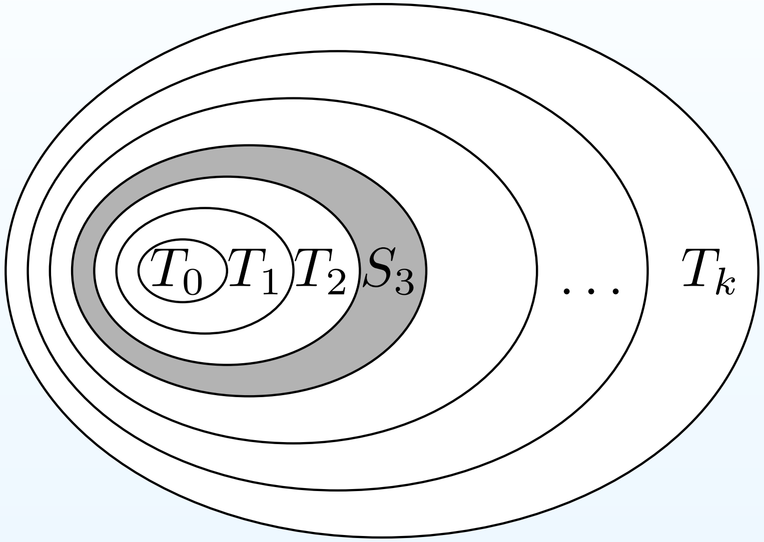

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- to succeed, much amplitude has to be in higher subspaces

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- [Klauck, Š, de Wolf, FOCS 2004]
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  $$T^2 S = \tilde{\Theta}(N^3)$$
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Evaluating Solutions to Systems of Linear Inequalities

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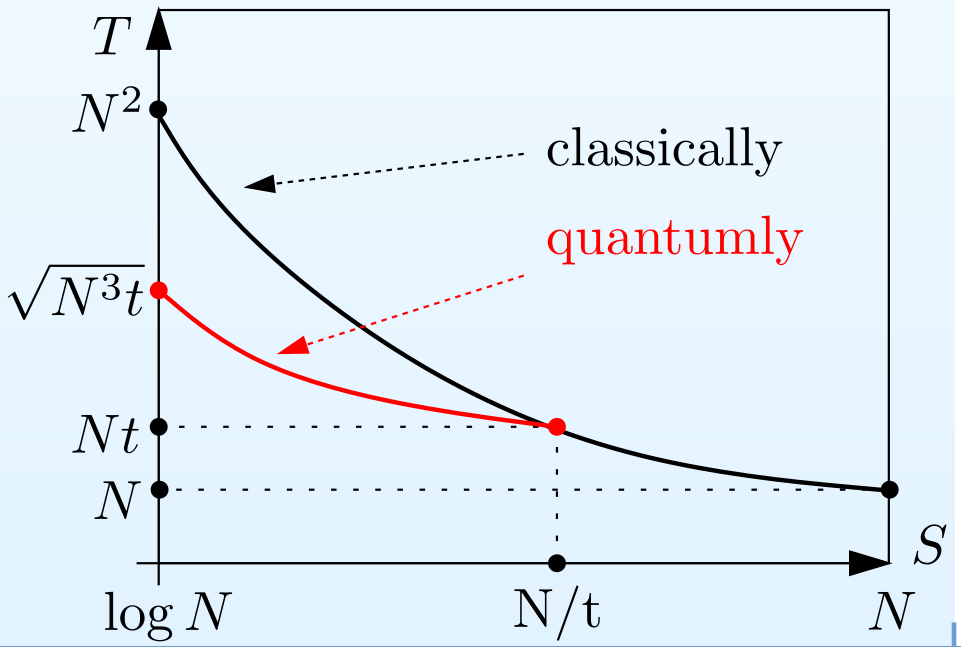

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$\Longrightarrow$ (tight) lower bound on T as a function of S

Summary and open problems

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Open problems

- binary AND-OR tree: $O(n^{0.753})$, $\Omega(\sqrt{n})$
- triangle finding: $O(n^{1.3})$, $\Omega(n)$
- verification of matrix products: $O(n^{5/3})$, $\Omega(n^{3/2})$