The codegree threshold of K_4^-

Jan Volec

MSCA global fellow at Emory University & Universität Hamburg

jointly with Victor Falgas-Ravry, Oleg Pighurko and Emil Vaughan

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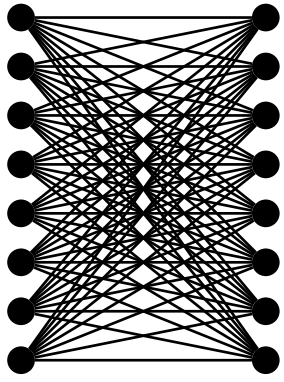

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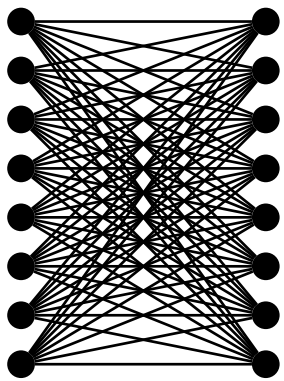

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Theorem (Turán, 1941)
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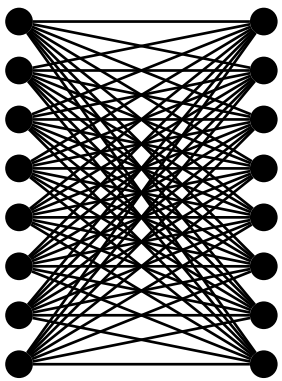

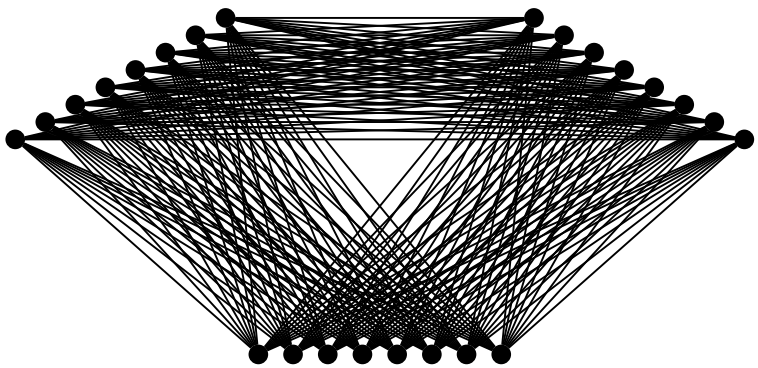

Extremal 3-uniform hypergraph (3-graph) theory

Turán's Theorem: Any n-vertex $G \not\supseteq K_k$ has $\leq \frac{k-2}{k-1} \cdot \frac{n^2}{2}$ edges.

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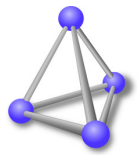
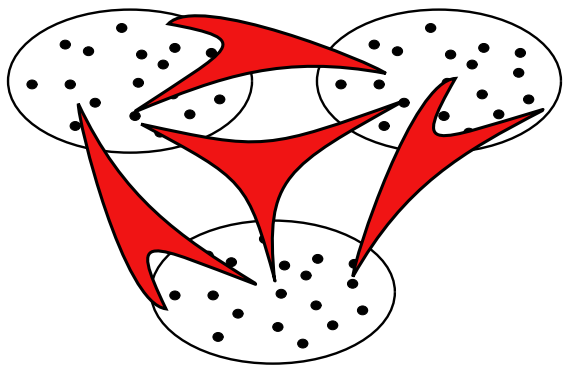


Conjecture: $\max \# \approx \frac{5}{9} \times \binom{n}{3}$



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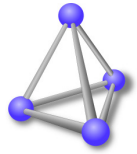

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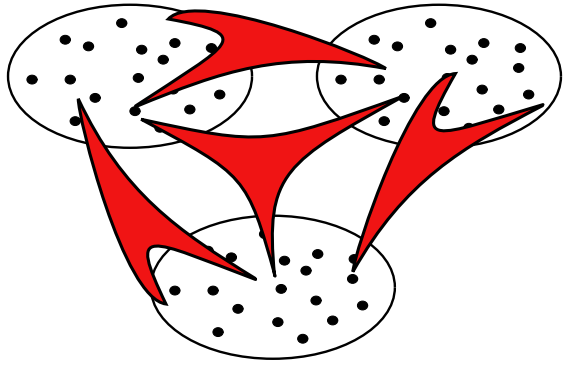

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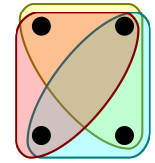

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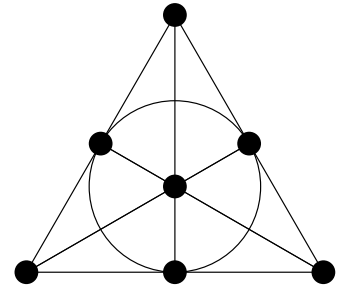

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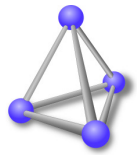

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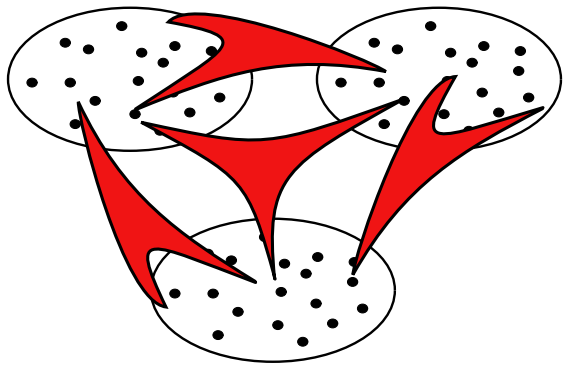

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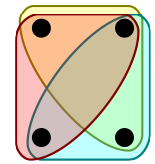

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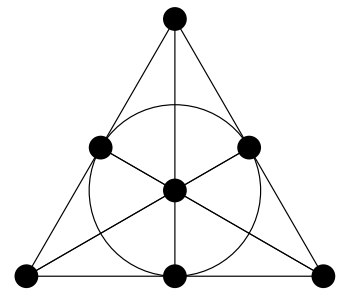

Turán density of $F \equiv$ minimum degree threshold for F

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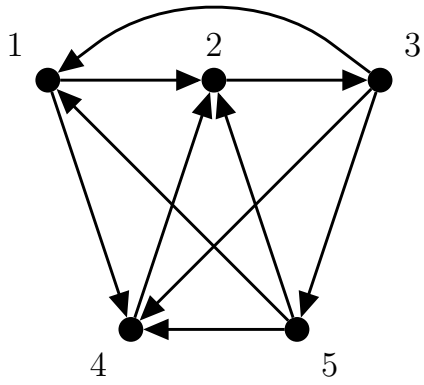

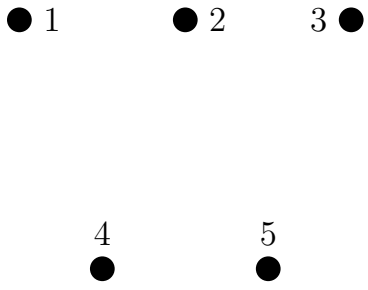

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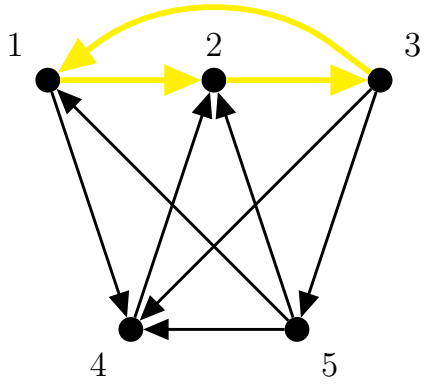

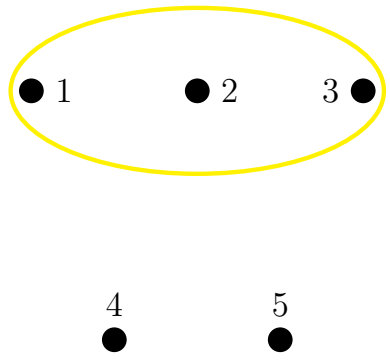

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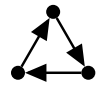 $\longleftrightarrow$ edges of 3-graph

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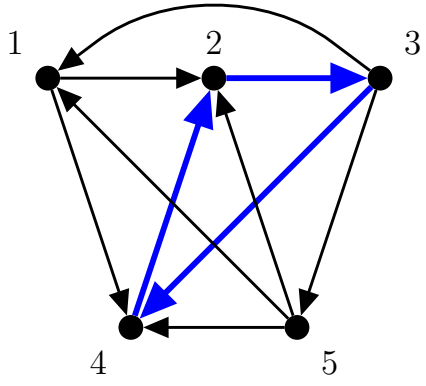

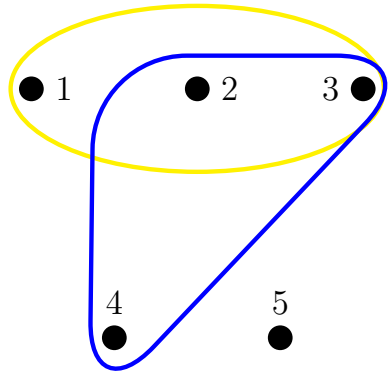

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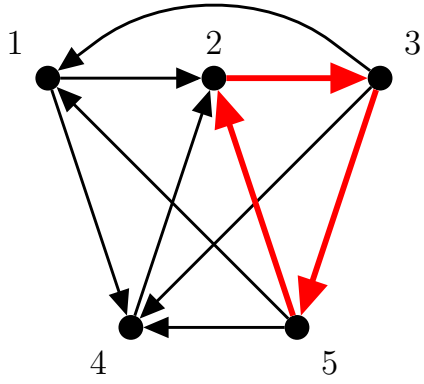

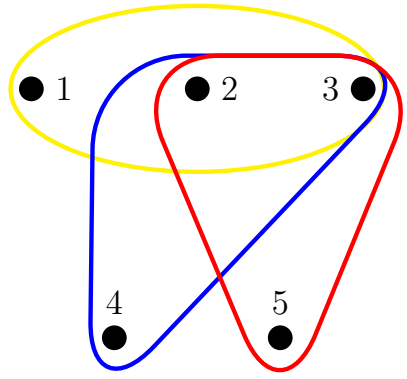

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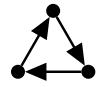 $\longleftrightarrow$ edges of 3-graph

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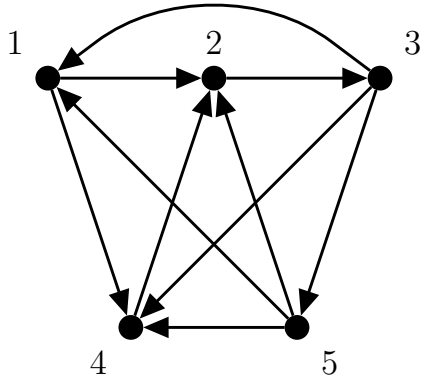

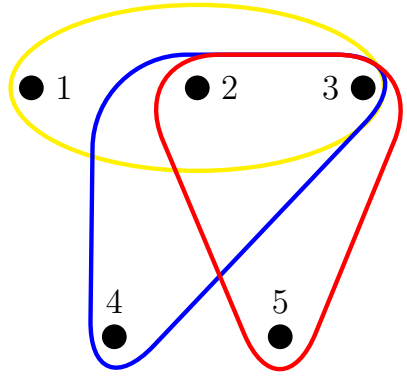

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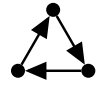
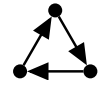
 $\longleftrightarrow$ edges of 3-graph

all u, v: $\approx \frac{n}{4}$ vertices give 

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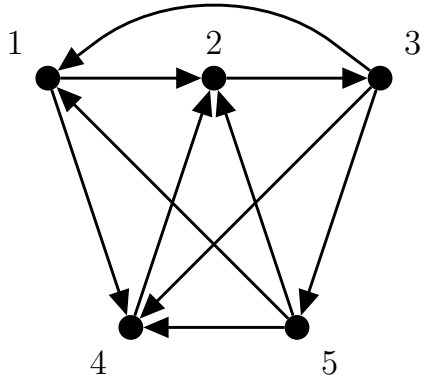

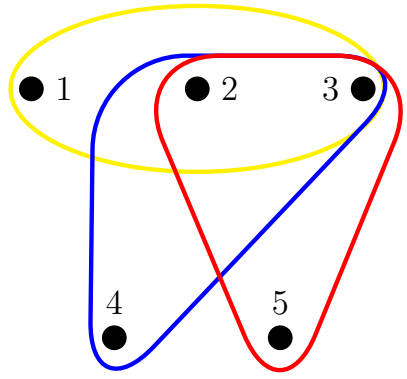

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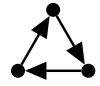 $\longleftrightarrow$ edges of 3-graph

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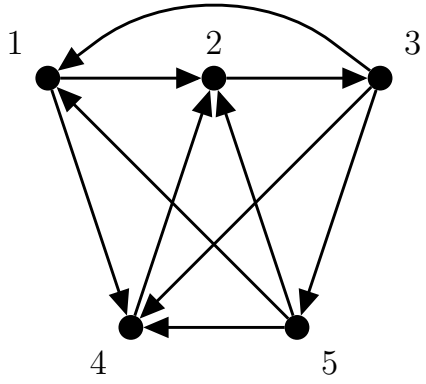

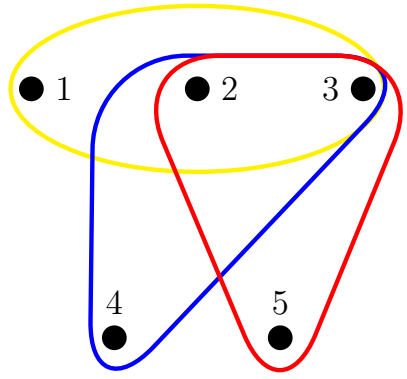

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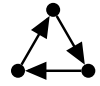 $\longleftrightarrow$ edges of 3-graph

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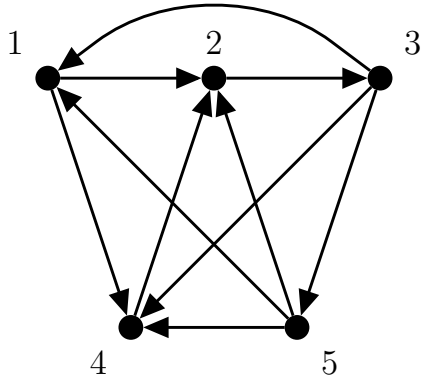

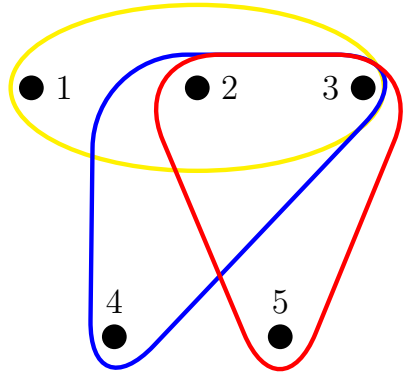

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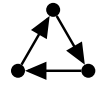 $\longleftrightarrow$ edges of 3-graph

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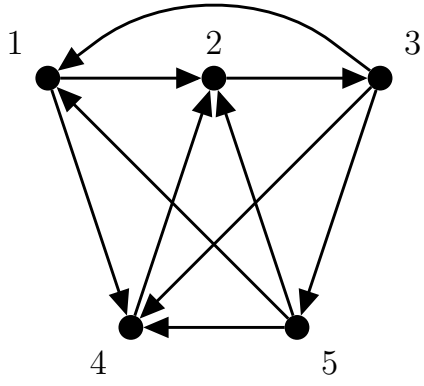

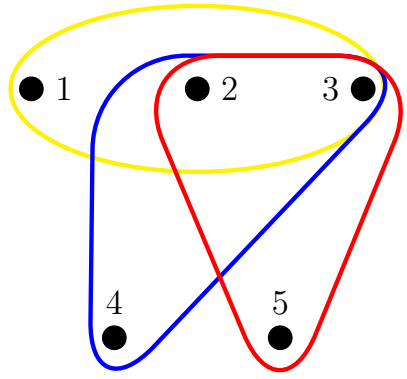

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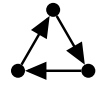 $\longleftrightarrow$ edges of 3-graph

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Main Theorem: The conjecture is true, $\mathrm{ex}_2(n, K_4^-) = \frac{n}{4} + O(1)$.

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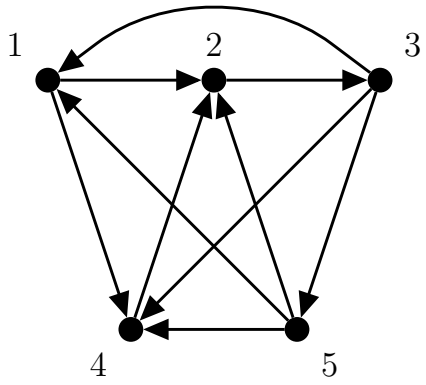

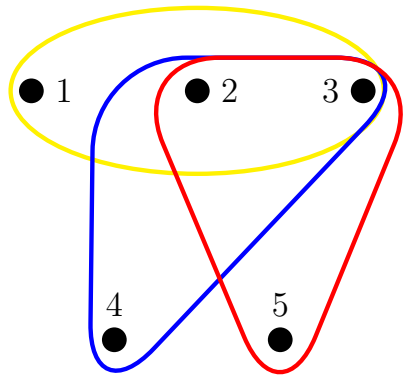

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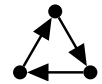
 $\longleftrightarrow$ edges of 3-graph

all u, v: $\approx \frac{n}{4}$ vertices give 

$\mathrm{codeg}(u,v) > \frac{n}{4} - o(n)$ whp

4 vertices can have ≤ 2 edges

$\mathrm{ex}_2(n, K_4^-) := \max_H \min_{u,v \in V(H)} \mathrm{codeg}(u,v)$ $\qquad$ H is n-vertex K_4^--free

Main Theorem: The conjecture is true, $\mathrm{ex}_2(n, K_4^-) = \frac{n}{4} + O(1)$.

Every K_4^--free H with $\mathrm{codeg}(u,v) > \frac{n}{4} - o(n)$ is "close" to EHF.

Exact threshold for K_4^- vs. skew Hadamard matrices

Main theorem$^+$: $\mathrm{ex}_2(n, K_4^-) \leq \left\lfloor \frac{n+1}{4} \right\rfloor$ for n large enough.

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Skew Hadamard matrix A of order ℓ: $AA^T = \ell I_\ell$ and $A + A^T = 2I_\ell$

$$\begin{pmatrix}
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-1 & +1 & +1 & +1 & -1 & +1 & -1 & -1 \\
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-1 & -1 & -1 & +1 & +1 & +1 & -1 & +1 \\
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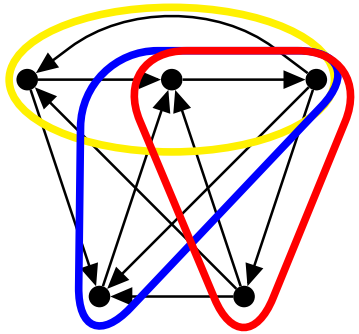
Skew Hadamard matrices known to exist for $4k = \ell < 188$, and

$$\ell = 2^t \prod_{q \in Q} (q+1) \text{ where } Q = \{q : q = p^a \wedge q \equiv 1 \mod 4\}$$

Seberry's conjecture on skew Hadamard matrices

$$\begin{pmatrix} +1 & +1 & +1 & +1 & +1 & +1 & +1 & +1 \\ -1 & +1 & +1 & +1 & -1 & +1 & -1 & -1 \\ -1 & -1 & +1 & +1 & +1 & -1 & +1 & -1 \\ -1 & -1 & -1 & +1 & +1 & +1 & -1 & +1 \\ -1 & +1 & -1 & -1 & +1 & +1 & +1 & -1 \\ -1 & -1 & +1 & -1 & -1 & +1 & +1 & +1 \\ -1 & +1 & -1 & +1 & -1 & -1 & +1 & +1 \\ -1 & +1 & +1 & -1 & +1 & -1 & -1 & +1 \end{pmatrix} \longrightarrow$$



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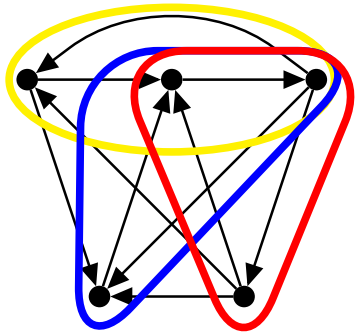

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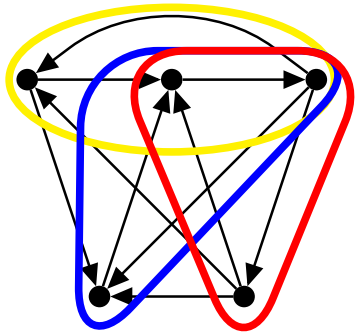

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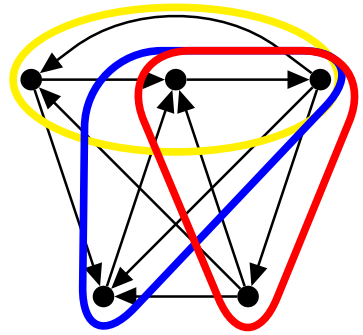

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Theorem Skew Hadamard $\exists$ for $4k \iff \mathrm{ex}_2(4k-1, K_4^-)$ is tight

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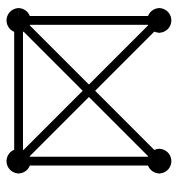
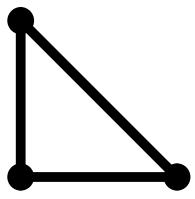
$G_k =$  $F =$ 

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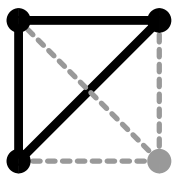
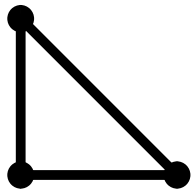
$G_k =$  $F =$ 

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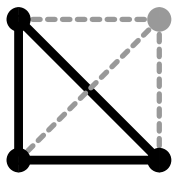
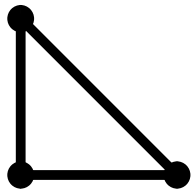
$G_k =$  $F =$ 

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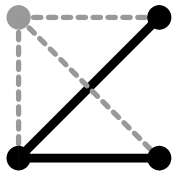
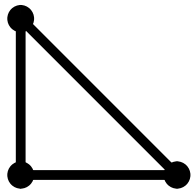
$G_k =$  $F =$ 

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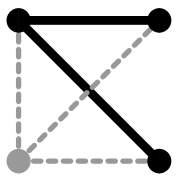
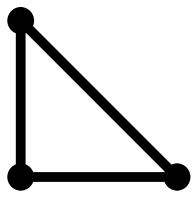
$G_k =$  $F =$ 

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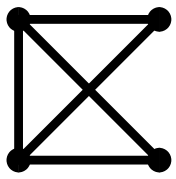
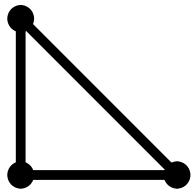
$G_k =$  $F =$  $\Longrightarrow p_k(F) = \frac{1}{2}$

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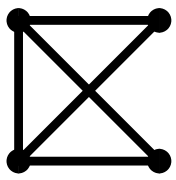
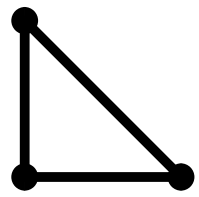
$G_k =$  $F =$  $\Longrightarrow p_k(F) = \frac{1}{2}$

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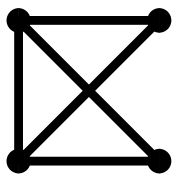
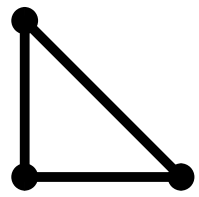
$G_k =$  $F =$  $\Longrightarrow p_k(F) = \frac{1}{2}$

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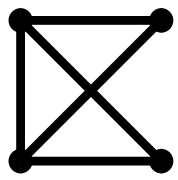
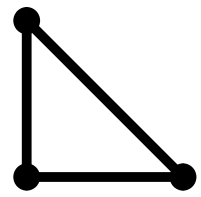
$G_k =$  $F =$  $\Longrightarrow p_k(F) = \frac{1}{2}$

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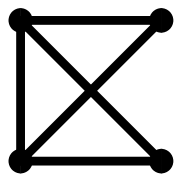
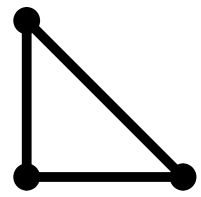
$G_k =$  $F =$  $\Longrightarrow p_k(F) = \frac{1}{2}$

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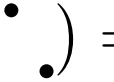
  Suppose $q($  $) = q($  $) = \frac{1}{2}$

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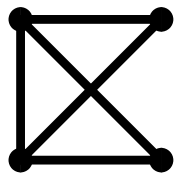
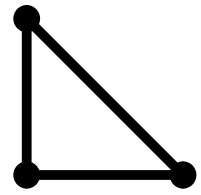
$G_k =$  $F =$  $\Longrightarrow p_k(F) = \frac{1}{2}$

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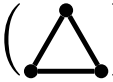
  Suppose $q($  $) = q($  $) = \frac{1}{2} \Longrightarrow$ other 3-vertex F: $q(F) = 0$

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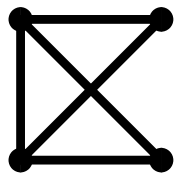
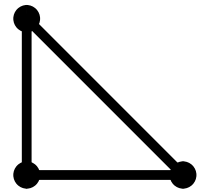
$G_k =$  $F =$  $\Longrightarrow p_k(F) = \frac{1}{2}$

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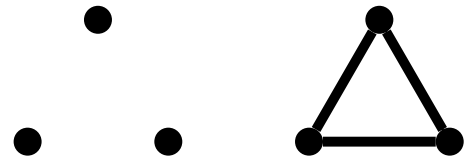

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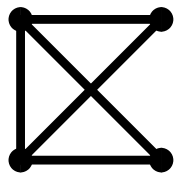
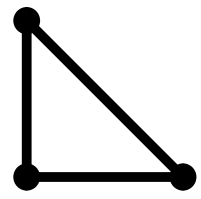
$G_k =$  $F =$  $\Longrightarrow p_k(F) = \frac{1}{2}$

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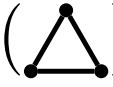
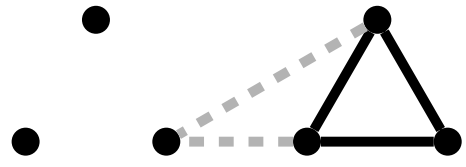
  Suppose $q($  $) = q($  $) = \frac{1}{2} \Longrightarrow$ other 3-vertex F: $q(F) = 0$



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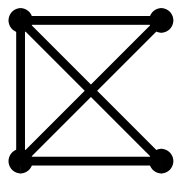
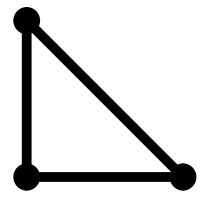
$G_k =$  $F =$  $\Longrightarrow p_k(F) = \frac{1}{2}$

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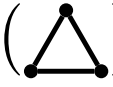
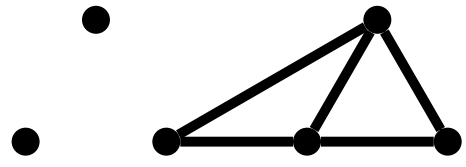
  Suppose $q($  $) = q($  $) = \frac{1}{2} \Longrightarrow$ other 3-vertex F: $q(F) = 0$



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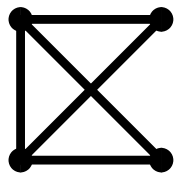
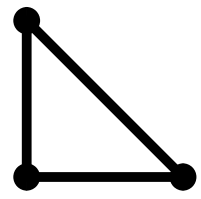
$G_k =$  $F =$  $\Longrightarrow p_k(F) = \frac{1}{2}$

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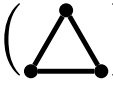
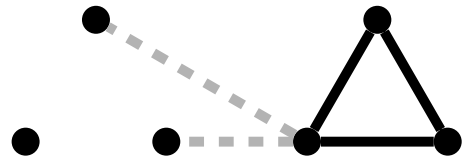
  Suppose $q($  $) = q($  $) = \frac{1}{2} \Longrightarrow$ other 3-vertex F: $q(F) = 0$



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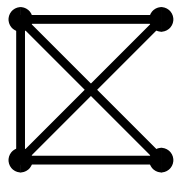
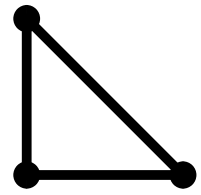
$G_k =$  $F =$  $\Longrightarrow p_k(F) = \frac{1}{2}$

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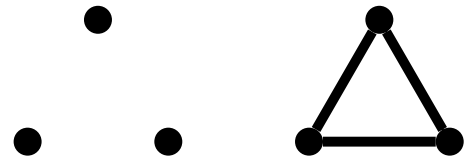

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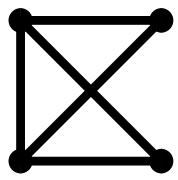
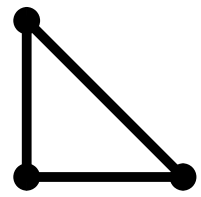
$G_k =$  $F =$  $\Longrightarrow p_k(F) = \frac{1}{2}$

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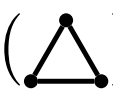
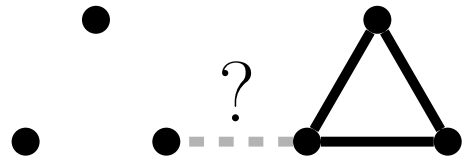
  Suppose $q($  $) = q($  $) = \frac{1}{2} \Longrightarrow$ other 3-vertex F: $q(F) = 0$



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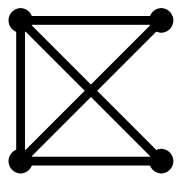
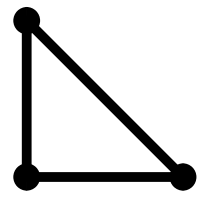
$G_k =$  $F =$  $\Longrightarrow p_k(F) = \frac{1}{2}$

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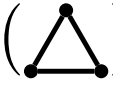
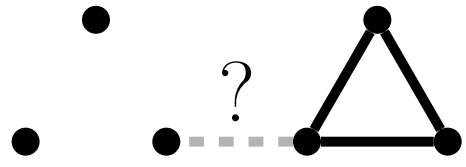
  Suppose $q($  $) = q($  $) = \frac{1}{2} \Longrightarrow$ other 3-vertex F: $q(F) = 0$



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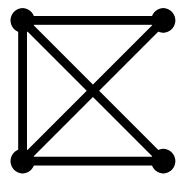
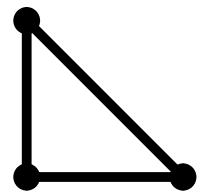
$G_k =$  $F =$  $\Longrightarrow p_k(F) = \frac{1}{2}$

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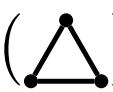
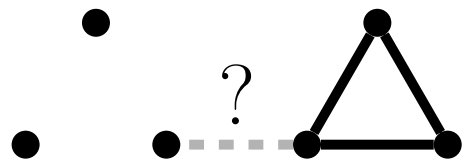
  Suppose $q($  $) = q($  $) = \frac{1}{2} \Longrightarrow$ other 3-vertex F: $q(F) = 0$



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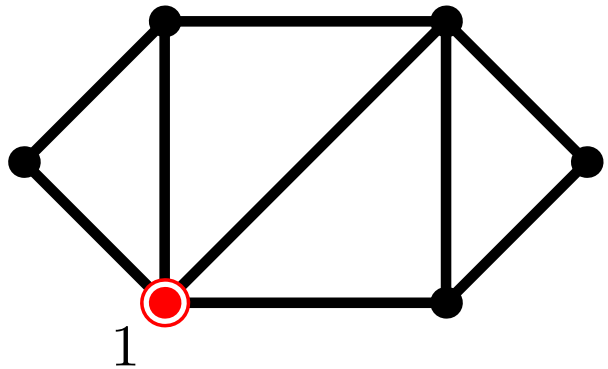
$G_k =$ 

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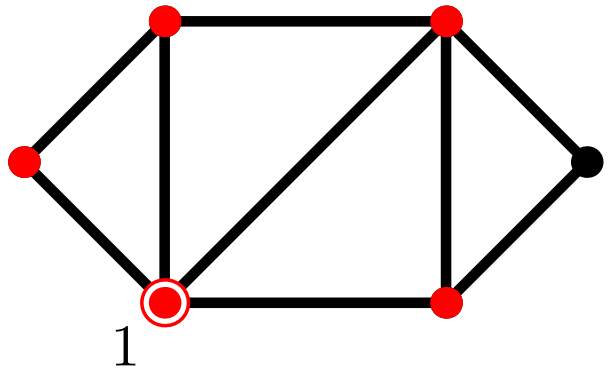
$G_k =$ 

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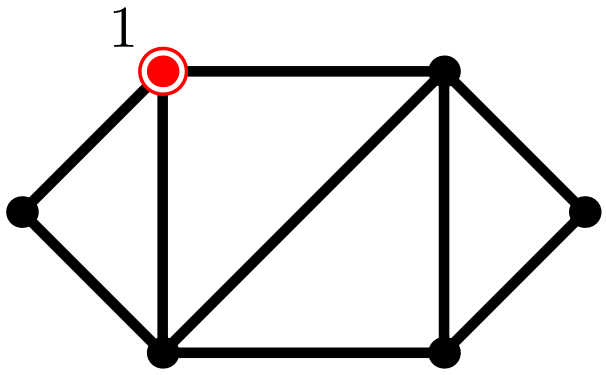
$G_k =$ 

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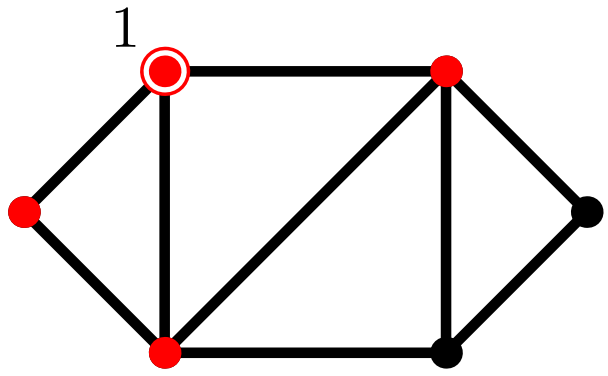
$G_k =$ 

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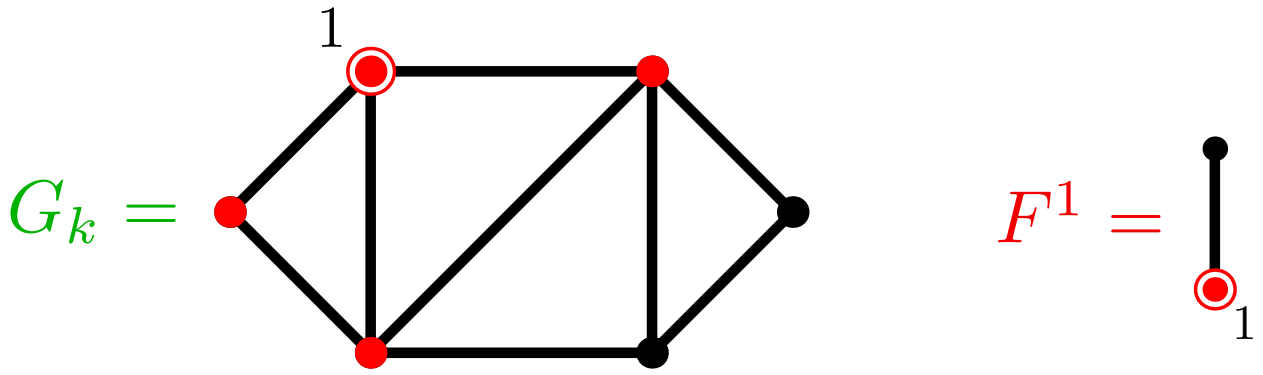

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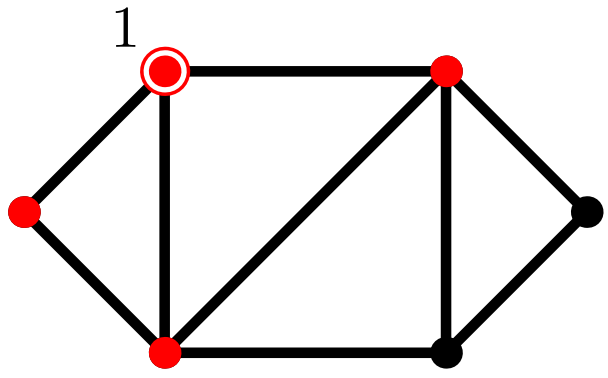
$G_k =$ 

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Flag Algebras – Rooted homomorphisms

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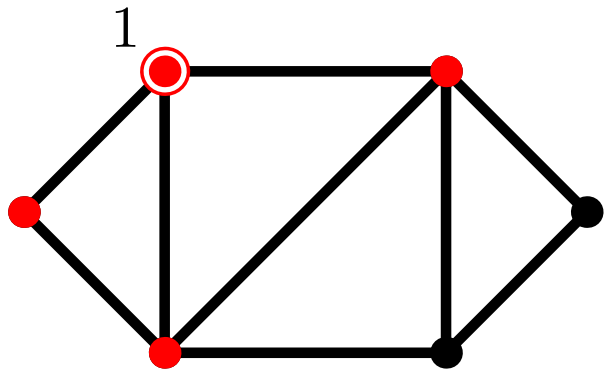
$G_k =$ 

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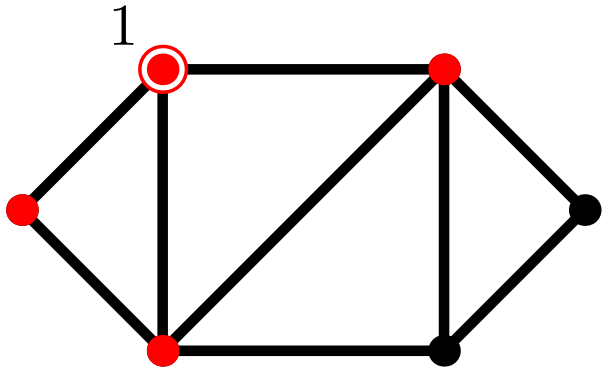
$G_k =$ 

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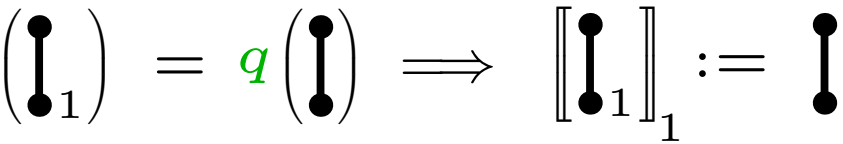

Flag Algebras – Averaging and Cauchy-Schwarz

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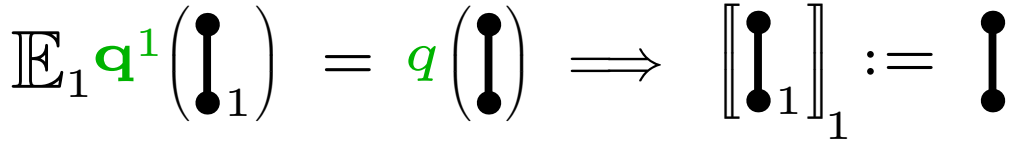

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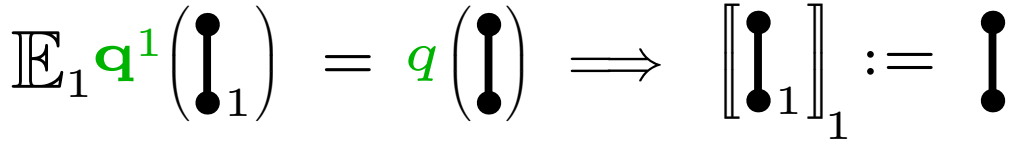

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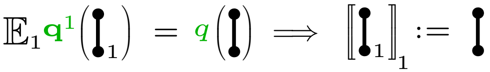

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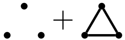
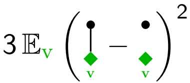
- Example — proof of Goodman's bound:  $\geq \frac{1}{4}$



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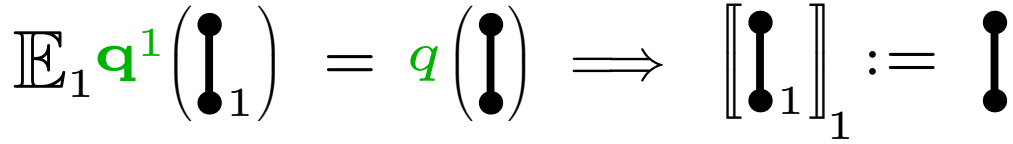

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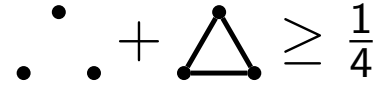

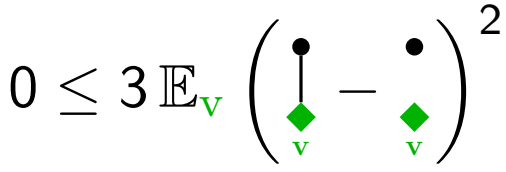

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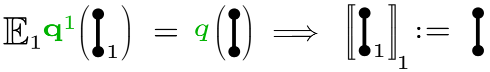

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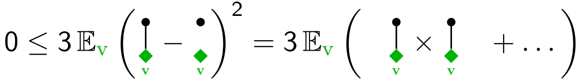

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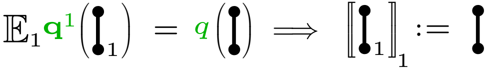

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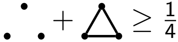
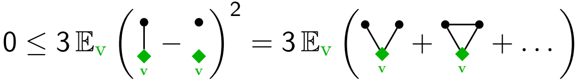
- Example — proof of Goodman's bound: 



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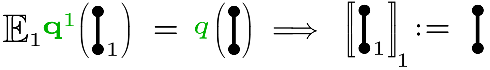

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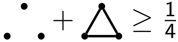
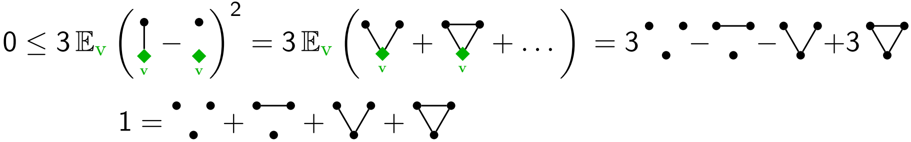
- Example — proof of Goodman's bound: 



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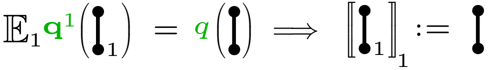

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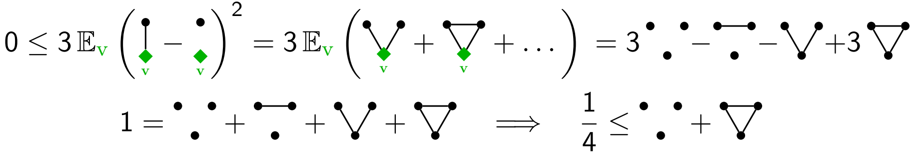

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$$1 = \;\; + \;\; + \;\; + \;\; \implies \frac{1}{4} \leq \;\; + \;\;$$

Semidefinite method – tool from Flag Algebra framework

Powerful tool, shed light to many long standing open problems

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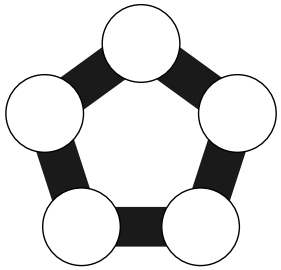
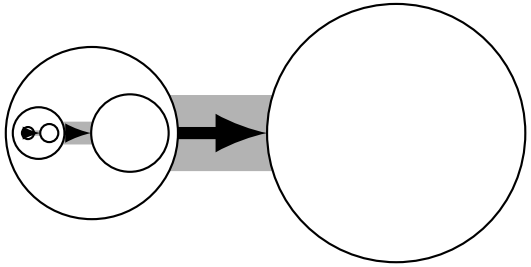
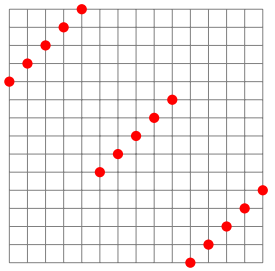
max  subj. to  $= 0$ max  in orient graph min $\overset{1234}{} + \overset{4321}{}$ in perm







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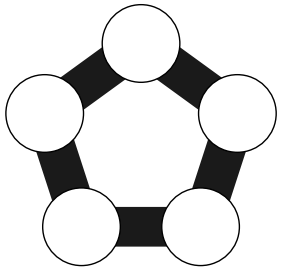

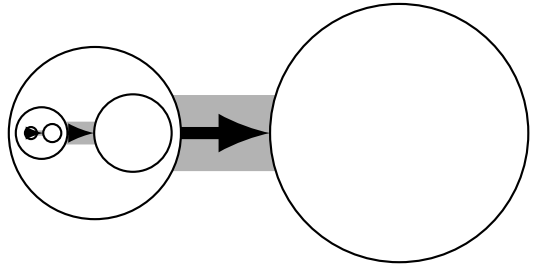

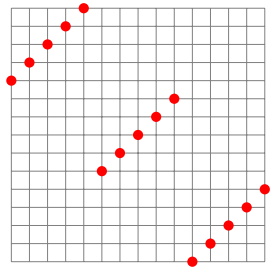

Typical instance $\max_{G \text{ large}} p(H, G)$ subject to constraints on G

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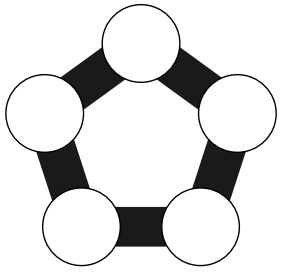

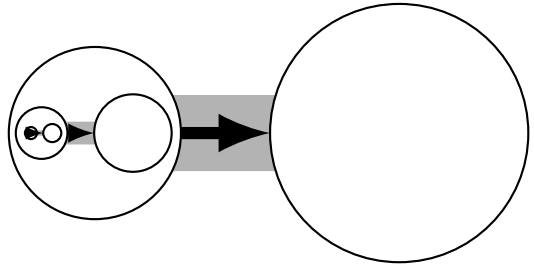

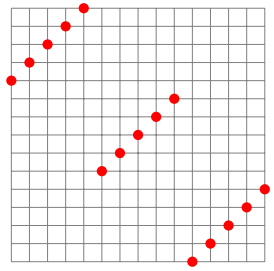

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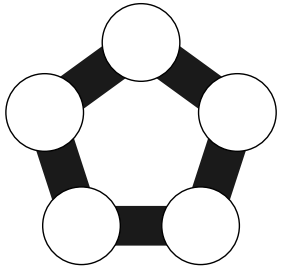

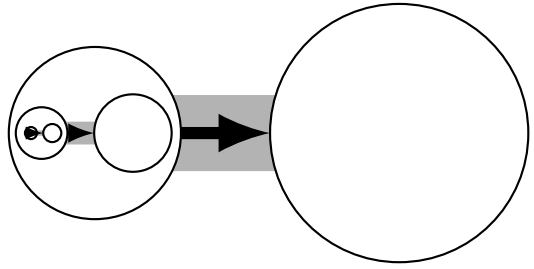

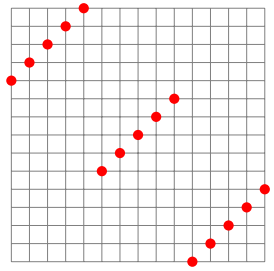

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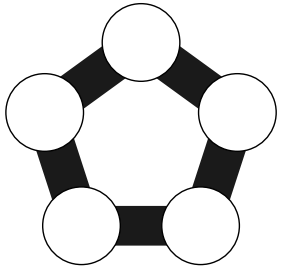

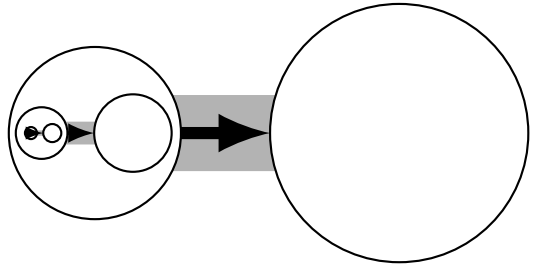

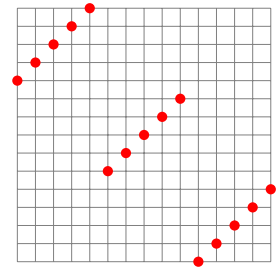

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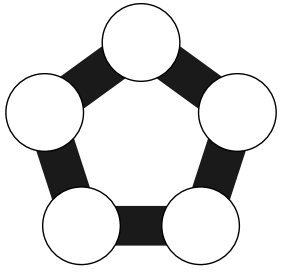

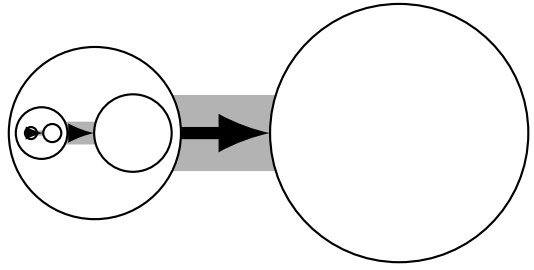

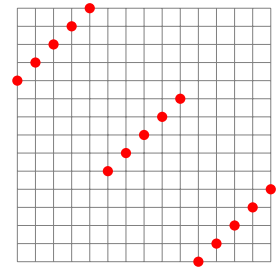

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Duality find specific Cauchy-Schwarz inequalities $\rightarrow$ give certificate

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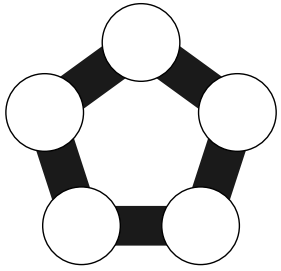
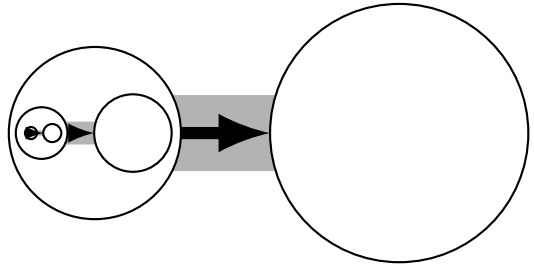
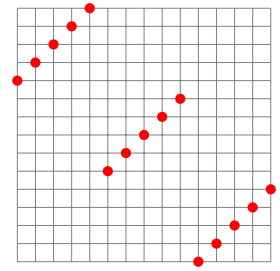






Typical instance $\max_{q \in \mathrm{LIM}} q(H)$ subject to constraints on q

where $\mathrm{LIM} = \{\text{homomorphisms } q : q(F) \geq 0\}$

Hatami-Norin: Given $Z = \sum \alpha_F \times F$, $\exists q$: $q(Z) < 0$ is undecidable

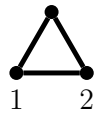
Idea relaxation, optimize over superset of LIM $\rightarrow$ semidefinite prg

Duality find specific Cauchy-Schwarz inequalities $\rightarrow$ give certificate

Such search: automatized and computer assisted (SDP solvers)

Flag algebras for the K_4^- codegree threshold in 3-graphs

AIM: $\mathrm{ex}_2(n, K_4^-) = \frac{n}{4} + O(1)$

Codegree assumption:  $- \frac{1}{4} \geq 0$ for every choice of $\{1, 2\} \subseteq V$

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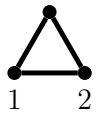
Codegree assumption: 

$- \frac{1}{4} \geq 0$ for every choice of $\{1, 2\} \subseteq V$

Use computer / SDP solver to prove identity for K_4^--free 3-graphs:

$$\left[\!\!\left[\left(\text{△}_{1\ 2} - \frac{1}{4} \right) \times b_H \cdot H \right]\!\!\right]_{12} + \sum_F c_F \cdot F + \left[\!\!\left[v_\sigma^{\mathrm{T}} D v_\sigma \right]\!\!\right]_\sigma = \text{⋰} - 3 \times \text{△}$$

for some $b_H \geq 0$, $c_F \geq 0$, and $D \succeq 0$ positive semidefinite matrix

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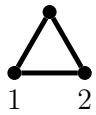
Codegree assumption:  $- \frac{1}{4} \geq 0$ for every choice of $\{1,2\} \subseteq V$

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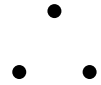
$$\left[\!\!\left[\left(\text{  } - \frac{1}{4} \right) \times b_H \cdot H \right]\!\!\right]_{12} + \sum_F c_F \cdot F + \left[\!\!\left[v_\sigma^{\mathrm{T}} D v_\sigma \right]\!\!\right]_\sigma = \text{  } - 3 \times \text{  }$$

for some $b_H \geq 0$, $c_F \geq 0$, and $D \succeq 0$ positive semidefinite matrix

$$\Longrightarrow 0 \leq \text{} - 3 \times \text{  } \iff \text{  } \leq \frac{1}{4}$$

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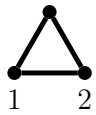
Codegree assumption:  $- \frac{1}{4} \geq 0$ for every choice of $\{1,2\} \subseteq V$

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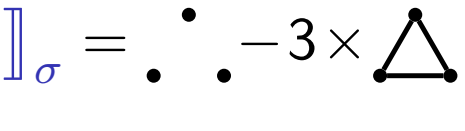
$$\left[\!\!\left[\left(\text{  } - \frac{1}{4} \right) \times b_H \cdot H \right]\!\!\right]_{12} + \sum_F c_F \cdot F + \left[\!\!\left[v_\sigma^{\mathrm{T}} D v_\sigma \right]\!\!\right]_\sigma = \text{  }$$

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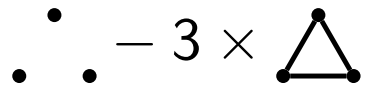
$$\implies 0 \leq \text{  } \iff \text{  } \leq \frac{1}{4}$$

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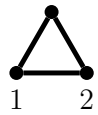
Codegree assumption:  $- \frac{1}{4} \geq 0$ for every choice of $\{1,2\} \subseteq V$

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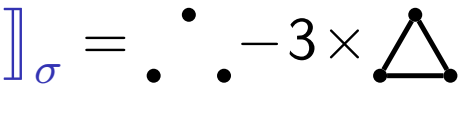
$$\left[\!\!\left[\left(\text{  } - \frac{1}{4} \right) \times b_H \cdot H \right]\!\!\right]_{12} + \sum_F c_F \cdot F + \left[\!\!\left[v_\sigma^{\mathrm{T}} D v_\sigma \right]\!\!\right]_\sigma = \text{  }$$

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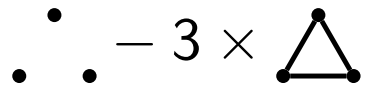
$$\implies 0 \leq \text{  } \iff \text{  } \leq \frac{1}{4}$$

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for some $b_H \geq 0$, $c_F \geq 0$, and $D \succeq 0$ positive semidefinite matrix

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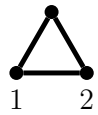
Codegree assumption:  $- \frac{1}{4} \geq 0$ for every choice of $\{1,2\} \subseteq V$

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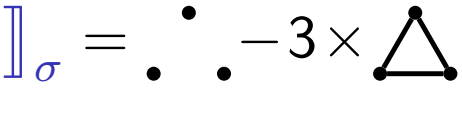
$$\left[\!\!\left[\left(\text{  } - \frac{1}{4} \right) \times b_H \cdot H \right]\!\!\right]_{12} + \sum_F c_F \cdot F + \left[\!\!\left[v_\sigma^{\mathrm{T}} D v_\sigma \right]\!\!\right]_\sigma = \text{  }$$

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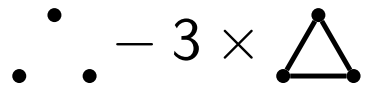
$$\implies 0 \leq \text{  } \iff \text{  } \leq \frac{1}{4}$$

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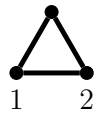
Codegree assumption:  $- \frac{1}{4} \geq 0$ for every choice of $\{1,2\} \subseteq V$

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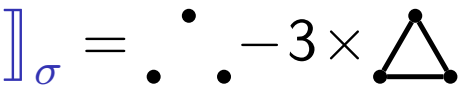
$$\left[\!\!\left[\left(\text{  } - \frac{1}{4} \right) \times b_H \cdot H \right]\!\!\right]_{12} + \sum_F c_F \cdot F + \left[\!\!\left[v_\sigma^{\mathrm{T}} D v_\sigma \right]\!\!\right]_\sigma = \text{  }$$

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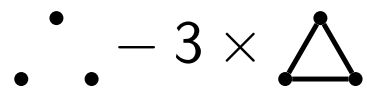
$$\implies 0 \leq \text{  } \iff \text{  } \leq \frac{1}{4}$$

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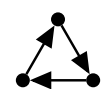
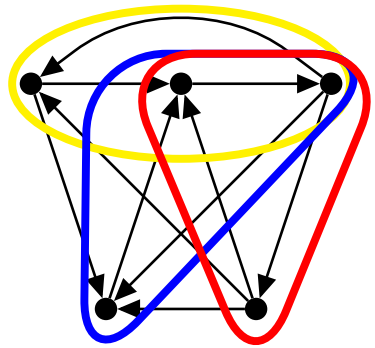
- $\implies$ we can construct auxiliary T_G s.t. $E(H) \approx$  (T_G)
- by codegree assumption T_G is quasi-random, hence $H \approx$ EHF

Conclusion

Theorem: $\mathrm{ex}_2(n, K_4^-) \leq \left\lfloor \frac{n+1}{4} \right\rfloor$ for n large enough.
$\mathrm{ex}_2(4k-1, K_4^-) = k \iff$ skew Hadamard $\exists$ for $4k$



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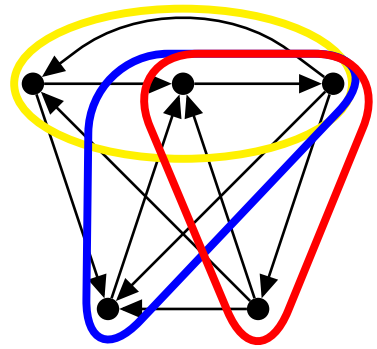

Conjecture: $\mathrm{ex}_2(n, K_4) = \frac{n}{2} + o(n)$?

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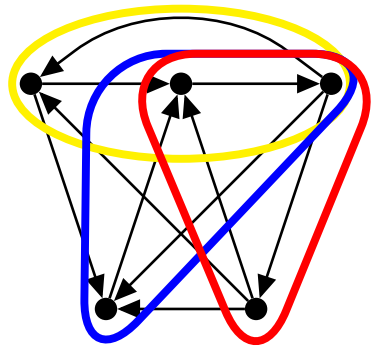

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When does d-weakly-QR H contain F? $e(H[S]) \approx d \cdot \binom{|S|}{3}$ $\forall S \subseteq V$

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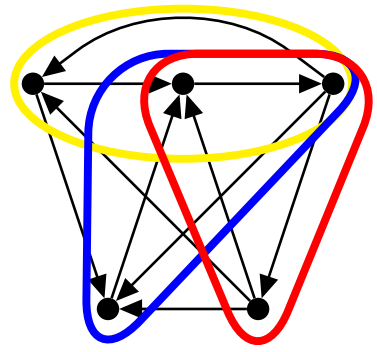

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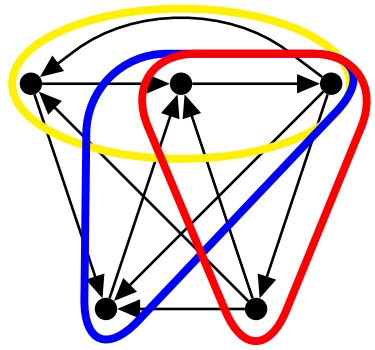

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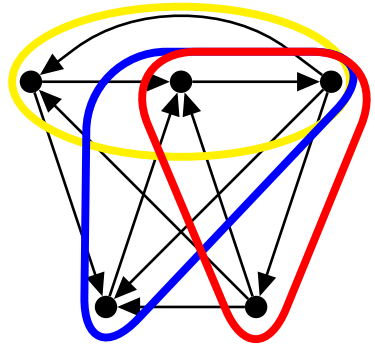

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Question: Is $\lim_{n\to\infty} \frac{1}{n} \cdot \mathrm{ex}_2(F) \geq \mathrm{ex}_{\mathrm{wqr}}(F)$ for every F?

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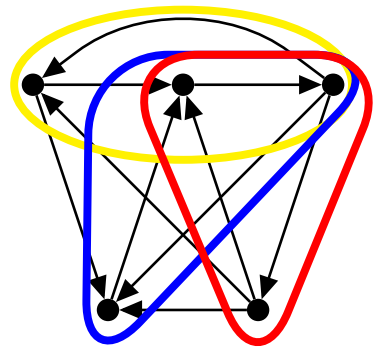

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Question$^+$: H is d-wqr $\Rightarrow$ large $S \subset V$: $\forall$ codeg $\ge d|S|$ in $H[S]$?

Conclusion

Thank you for your attention!

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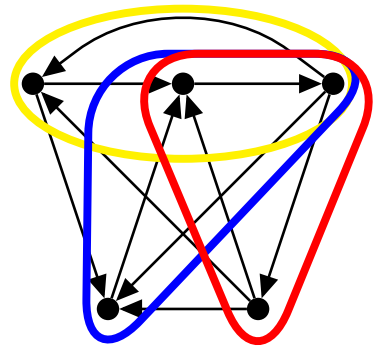

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