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STRUCTURAL RAMSEY THEORY AND THE
EXTENSION PROPERTY FOR PARTIAL
AUTOMORPHISMS

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P R E F A C E

This habilitation thesis consists of one book chapter, five papers (published or accepted of publication) and three yet unpublished manuscripts listed below. All these works aim to develop a general theory dealing with two properties of classes of finite structures

1. the Ramsey property and
2. the Extension Property for Partial Automorphisms (EPPA).

The Introduction presents a synthesis of these results and puts them into the context of current research emphasizing the convergence of these two areas which have been until recently developed independently.

[HN15] Jan Hubička and Jaroslav Nešetřil. Universal structures with forbidden homomorphisms. In Å. Hirvonen, J. Kontinen, R. Kosak, and A. Villaveces, editors, *Logic Without Borders: Essays on Set Theory, Model Theory, Philosophical Logic and Philosophy of Mathematics*, Ontos Mathematical Logic Series, pages 241–264. De Gruyter, 2015.

<https://doi.org/10.1515/9781614516873.241>

[HN18A] Jan Hubička and Jaroslav Nešetřil. Bowtie-free graphs have a Ramsey lift. *Advances in Applied Mathematics*, 96:286 – 311, 2018.

<https://doi.org/10.1016/j.aam.2017.12.005>

[HN19] Jan Hubička and Jaroslav Nešetřil. All those Ramsey classes (Ramsey classes with closures and forbidden homomorphisms). *Advances in Mathematics*, 356C:106791, 2019.

<https://doi.org/10.1016/j.aim.2019.106791>

[EHN17] David M. Evans, Jan Hubička, and Jaroslav Nešetřil. Ramsey properties and extending partial automorphisms for classes of finite structures. Submitted, arXiv:1705.02379, 2017.

<https://arxiv.org/abs/1705.02379>

[EHN19] David M. Evans, Jan Hubička, and Jaroslav Nešetřil. Automorphism groups and Ramsey properties of sparse graphs. *Proceedings of the London Mathematical Society*, 119(2):515–546, 2019.

<https://doi.org/10.1112/plms.12238>

[ABWH⁺17B] Andres Aranda, David Bradley-Williams, Jan Hubička, Miltiadis Karamanlis, Michael Kompatscher, Matěj Konečný, and

Micheal Pawliuk. Ramsey expansions of metrically homogeneous graphs. Submitted, arXiv:1707.02612, 2017.

<https://arxiv.org/abs/1707.02612>

[HKN19A] Jan Hubička, Matěj Konečný, and Jaroslav Nešetřil. A combinatorial proof of the extension property for partial isometries. *Commentationes Mathematicae Universitatis Carolinae*, 60(1):39–47, 2019.

<https://doi.org/10.14712/1213-7243.2015.275>

[HKN19B] Jan Hubička, Matěj Konečný, and Jaroslav Nešetřil. All those EPPA classes (strengthenings of the Herwig-Lascar theorem). arXiv:1902.03855, 2019.

<https://arxiv.org/abs/1902.03855>

[EHKN19] David M. Evans, Jan Hubička, Matěj Konečný, and Jaroslav Nešetřil. EPPA for two-graphs and antipodal metric spaces. *Proceedings of the American Mathematical Society*, 2019. In print, arXiv:1812.11157.

<https://arxiv.org/abs/1812.11157>

While not included directly in this habilitation, additionally six publications [HN05, HN16, HKN17, HN17, ABWH⁺17a, HN18b] and two manuscripts [HKK18, EHKL19] I co-authored on this topic are discussed in the Introduction, too.

Prague, September 30, 2019

Jan Hubička

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INTRODUCTION

In this chapter we survey recent developments concerning two properties of classes of finite structures (introduced formally in Section 1.1).

1. Class \mathcal{K} of finite structures is *Ramsey* (or has the *Ramsey property*) if for every $\mathbf{A}, \mathbf{B} \in \mathcal{K}$ there exists $\mathbf{C} \in \mathcal{K}$ such that for every colouring of the embeddings $\mathbf{A} \rightarrow \mathbf{C}$ with two colours there exists an embedding $f: \mathbf{B} \rightarrow \mathbf{C}$ such that all embeddings $\mathbf{A} \rightarrow f(\mathbf{B})$ have the same colour.

The study of Ramsey classes emerged as a generalisation of the classical results of Ramsey theory. Ramsey theorem itself implies that the class of all finite linearly ordered sets is Ramsey. The first true structural Ramsey theorem was given in 1977 by Nešetřil and Rödl [NR77] and, independently, by Abramson and Harrington [AH78]. By this result, the class of all finite graphs (or, more generally, relational structures in a given language) endowed with a linear ordering of vertices is Ramsey. The celebrated Nešetřil–Rödl theorem [NR77] states that the classes of relational structures endowed with a linear order and omitting a given set of irreducible substructures is Ramsey. Additional examples are discussed in Section 1.6.

2. Class \mathcal{K} of finite structures has the *extension property for partial automorphisms* (EPPA for short, sometimes also called the *Hrushovski property*) if for every $\mathbf{A} \in \mathcal{K}$ there exists $\mathbf{B} \in \mathcal{K}$ containing \mathbf{A} as an (induced) substructure such that every partial automorphism of \mathbf{A} (an isomorphism between two induced substructures of \mathbf{A}) extends to an automorphism of \mathbf{B} .

This concept was originally motivated by the *small index conjecture* from group theory which states that a subgroup of the automorphism group of the countable random graph is of countable index if and only if it contains a stabiliser of a finite set. This conjecture was proved by Hodges, Hodkinson, Lascar and Shelah in 1993 [HHLs93]. To complete the argument they used a result of Hrushovski [Hru92] who, in 1992, established that the class of all finite graphs has EPPA. After this, the quest of identifying new classes of structures with EPPA continued. Herwig generalised Hrushovski’s result to all finite relational structures [Her95] (in a fixed finite language) and later to the classes of all finite relational structures omitting certain sets of irreducible substructures [Her98]. He also developed several combinatorial strategies

to establish EPPA for a given class. This development culminated to the Herwig–Lascar theorem [HL00], one of the deepest results in the area, which established a structural condition for a class to have EPPA. Again many more examples of classes having EPPA are known today and are discussed in Section 1.6.

While EPPA and the Ramsey property may not seem related at first glance, they are both (under mild and natural assumptions) strengthenings of a model-theoretical notion of the amalgamation property (Definition 1.1.1). It was noticed by Nešetřil in late 1980’s [Neš89, Neš05] that every Ramsey class with the joint embedding property has the amalgamation property. The same holds for EPPA as well (in fact, EPPA was introduced as a stronger form of amalgamation).

By the classical Fraïssé theorem [Fra53] (Theorem 1.5.1), every amalgamation class \mathcal{K} (that is, an isomorphism-closed hereditary class of finite structures with the amalgamation property, the joint embedding, see Definition 1.1.2) with countably many mutually non-isomorphic structures has an up to isomorphism unique countable *Fraïssé limit* \mathbf{H} . The structure \mathbf{H} is *homogeneous*: every partial automorphism of \mathbf{H} with finite domain extends to an automorphism of \mathbf{H} . The correspondence between amalgamation classes and homogeneous structures is one-to-one, every amalgamation class with only countably many mutually non-isomorphic structures is precisely the class of all finite structures which embed to \mathbf{H} (the *age* of \mathbf{H}).

In a surprising development with major implications for the subject, in the period 2005–2008 both the Ramsey property and EPPA were shown to be intimately linked to topological dynamics. In 2005, Kechris, Pestov and Todorčević [KPT05] showed that the automorphism group of a homogeneous structure \mathbf{H} is extremely amenable if and only if its age is Ramsey. In 2008, Kechris and Rosendal [KR07] showed that EPPA implies amenability. This gave a fresh motivation to seek deeper understanding and additional examples of Ramsey and EPPA classes. We discuss some recent developments in this area with a focus on progress in systematising the combinatorial techniques on which the applications of the new theory ultimately are based. For further connections see, for example, Nguyen Van Thé’s [NVT15], Bodirsky’s [Bod15] and Solecky’s [Sol11] surveys (for the Ramsey context); Macpherson’s survey [Mac11] (for homogeneous structures) and Siniora’s thesis [Sin17b] (for the EPPA context).

1.1 PRELIMINARIES

1.1.1 Γ_L -structures

We find it convenient to work with model-theoretical structures (see, for example, [Hod93]) generalised in two ways.

In the first place, we equip the language with a permutation group. In the classical approach this group is trivial.

In the second place, we allow the language to include functions with values in the power set of the structure. This allows for an abstract treatment of the notion of algebraic closure (Definition 1.6.3).

This extended formalism was introduced by Hubička, Konečný and Nešetřil to greatly simplify the presentation of EPPA constructions and also to make it easy to speak about free amalgamation classes in languages containing non-unary functions [HKN19b] (see also [HN19] and [EHN17] which use a related formalism in the Ramsey context).

Let $L = L_{\mathcal{R}} \cup L_{\mathcal{F}}$ be a *language* with *relational symbols* $R \in L_{\mathcal{R}}$ and *function symbols* $F \in L_{\mathcal{F}}$ each having its *arity* denoted by $\text{arity}(R)$ for relations and $\text{arity}(F)$ for functions.

Let Γ_L be a permutation group on L which preserves types and arities of all symbols. We say that Γ_L is a *language equipped with a permutation group*.

Denote by $\mathcal{P}(A)$ the set of all subsets of A . A Γ_L -*structure* \mathbf{A} is a structure with *vertex set* A , functions $F_{\mathbf{A}} : A^{\text{arity}(F)} \rightarrow \mathcal{P}(A)$ for every $F \in L_{\mathcal{F}}$ and relations $R_{\mathbf{A}} \subseteq A^{\text{arity}(R)}$ for every $R \in L_{\mathcal{R}}$. (By $\mathcal{P}(A)$ we denote the power set of \mathbf{A} .) Notice that the domain of a function is ordered while the range is unordered.

If the permutation group Γ_L is trivial, we also speak of L -structures instead of Γ_L -structures. The permutation of language is motivated by applications in the context of EPPA. Presently there are no applications of this concept in the Ramsey context. For this reason we formulate many definitions and results for L -structures, only so that we can refer to published proofs in the area without the need to justify their validity for Γ_L -structures.

Note that we will still consider set-valued functions for L -structures. If, in addition, all value ranges of all functions consist of singletons, one obtains the usual notion of model-theoretical language and structures. All results and constructions in this thesis presented on Γ_L -structures and L -structures can thus be applied in this context.

If the vertex set A is finite, we call \mathbf{A} a *finite structure* and by $\text{Str}(\Gamma_L)$ we denote the class of all finite Γ_L -structures. We consider only structures with finitely or countably infinitely many vertices. If $L_{\mathcal{F}} = \emptyset$,

we call Γ_L a *relational language* and say that a Γ_L -structure is a *relational Γ_L -structure*. A function F such that $\text{arity}(F) = 1$ is a *unary function*.

1.1.2 Maps between Γ_L -structures

The standard notions of homomorphism, monomorphism and embedding extend to Γ_L -structures naturally as follows.

A *homomorphism* $f: \mathbf{A} \rightarrow \mathbf{B}$ is a pair $f = (f_L, f_A)$ where $f_L \in \Gamma_L$ and f_A is a mapping $A \rightarrow B$ such that:

(a) for every relation symbol $R \in L_{\mathcal{R}}$ of arity a it holds:

$$(x_1, x_2, \dots, x_a) \in R_{\mathbf{A}} \implies (f_A(x_1), f_A(x_2), \dots, f_A(x_a)) \in f_L(R)_{\mathbf{B}},$$

(b) for every function symbol $F \in L_{\mathcal{F}}$ of arity a it holds:

$$f_A(F_{\mathbf{A}}(x_1, x_2, \dots, x_a)) \subseteq f_L(F)_{\mathbf{B}}(f_A(x_1), f_A(x_2), \dots, f_A(x_a)).$$

For brevity we will also write $f(x)$ for $f_A(x)$ in the context where $x \in A$ and $f(S)$ for $f_L(S)$ where $S \in L$. For a subset $A' \subseteq A$ we denote by $f(A')$ the set $\{f(x) : x \in A'\}$ and by $f(\mathbf{A})$ the homomorphic image of a Γ_L -structure \mathbf{A} .

If f_A is injective then f is called a *monomorphism*. A monomorphism f is an *embedding* if:

(a) for every relation symbol $R \in L_{\mathcal{R}}$ of arity a it holds:

$$(x_1, x_2, \dots, x_a) \in R_{\mathbf{A}} \iff (f_A(x_1), f_A(x_2), \dots, f_A(x_a)) \in f_L(R)_{\mathbf{B}},$$

(b) for every function symbol $F \in L_{\mathcal{F}}$ of arity a it holds:

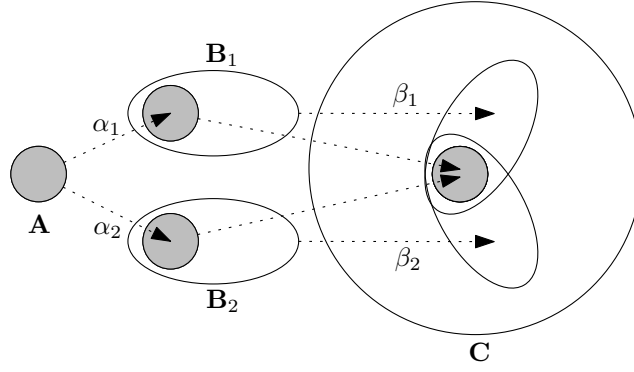
$$f_A(F_{\mathbf{A}}(x_1, x_2, \dots, x_a)) = f_L(F)_{\mathbf{B}}(f_A(x_1), f_A(x_2), \dots, f_A(x_a)).$$

If f is an embedding where f_A is one-to-one then f is an *isomorphism*. If f_A is an inclusion and f_L is the identity then \mathbf{A} is a *substructure* of \mathbf{B} and $f(\mathbf{A})$ is also called a *copy* of \mathbf{A} in \mathbf{B} . For an embedding $f: \mathbf{A} \rightarrow \mathbf{B}$ we say that \mathbf{A} is *isomorphic* to $f(\mathbf{A})$.

Given $\mathbf{A} \in \mathcal{K}$ and $B \subseteq A$, the *closure of B in \mathbf{A}* , denoted by $\text{Cl}_{\mathbf{A}}(B)$, is the smallest substructure of \mathbf{A} containing B . For $x \in A$ we will also write $\text{Cl}_{\mathbf{A}}(x)$ for $\text{Cl}_{\mathbf{A}}(\{x\})$.

1.1.3 Amalgamation classes, homogeneous structures and the Fraïssé theorem

Basically all classes of structures of interest in this work are the so-called amalgamation classes which we introduce now.

Figure 1.1: An amalgamation of \mathbf{B}_1 and \mathbf{B}_2 over \mathbf{A} .

Definition 1.1.1 (Amalgamation [Fra53]). Let \mathbf{A} , \mathbf{B}_1 and \mathbf{B}_2 be Γ_L -structures, α_1 an embedding of \mathbf{A} into \mathbf{B}_1 and α_2 an embedding of \mathbf{A} into \mathbf{B}_2 . Then every Γ_L -structure \mathbf{C} with embeddings $\beta_1: \mathbf{B}_1 \rightarrow \mathbf{C}$ and $\beta_2: \mathbf{B}_2 \rightarrow \mathbf{C}$ such that $\beta_1 \circ \alpha_1 = \beta_2 \circ \alpha_2$ (note that this must also hold for the language part of α_i 's and β_i 's) is called an *amalgamation* of \mathbf{B}_1 and \mathbf{B}_2 over \mathbf{A} with respect to α_1 and α_2 .

See Figure 1.1. We will often call \mathbf{C} simply an *amalgamation* of \mathbf{B}_1 and \mathbf{B}_2 over \mathbf{A} (in most cases α_1 and α_2 can be chosen to be inclusion embeddings). If the structure \mathbf{A} is empty, we call \mathbf{C} the *joint embedding* of \mathbf{B}_1 and \mathbf{B}_2 .

We say that the amalgamation is *strong* if it holds that $\beta_1(x_1) = \beta_2(x_2)$ if and only if $x_1 \in \alpha_1(A)$ and $x_2 \in \alpha_2(A)$. Strong amalgamation is *free* if $C = \beta_1(B_1) \cup \beta_2(B_2)$ and whenever a tuple \bar{x} of vertices of \mathbf{C} contains vertices of both $\beta_1(B_1 \setminus \alpha_1(A))$ and $\beta_2(B_2 \setminus \alpha_2(A))$, then \bar{x} is in no relation of \mathbf{C} , and for every function $F \in L$ with $\text{arity}(F) = |\bar{x}|$ it holds that $F_{\mathbf{C}}(\bar{x}) = \emptyset$.

Definition 1.1.2 (Amalgamation class [Fra53]). An *amalgamation class* is a class \mathcal{K} of finite Γ_L -structures which is closed for isomorphisms and satisfies the following three conditions:

1. *Hereditary property*: For every $\mathbf{A} \in \mathcal{K}$ and every structure \mathbf{B} with an embedding $f: \mathbf{B} \rightarrow \mathbf{A}$ we have $\mathbf{B} \in \mathcal{K}$;
2. *Joint embedding property*: For every $\mathbf{A}, \mathbf{B} \in \mathcal{K}$ there exists $\mathbf{C} \in \mathcal{K}$ with embeddings $f: \mathbf{A} \rightarrow \mathbf{C}$ and $g: \mathbf{B} \rightarrow \mathbf{C}$;
3. *Amalgamation property*: For $\mathbf{A}, \mathbf{B}_1, \mathbf{B}_2 \in \mathcal{K}$ and embeddings $\alpha_1: \mathbf{A} \rightarrow \mathbf{B}_1$ and $\alpha_2: \mathbf{A} \rightarrow \mathbf{B}_2$, there is $\mathbf{C} \in \mathcal{K}$ which is an amalgamation of \mathbf{B}_1 and \mathbf{B}_2 over \mathbf{A} with respect to α_1 and α_2 .

If the \mathbf{C} in the amalgamation property can always be chosen as the free amalgamation, then \mathcal{K} is a *free amalgamation class*. Analogously

if \mathbf{C} can be always chosen as a strong amalgamation, then \mathcal{K} is *strong amalgamation class*

A class \mathcal{K} of finite Γ_L has *joint embedding property* if for every $\mathbf{B}_1, \mathbf{B}_2 \in \mathcal{K}$ there exists a joint embedding $\mathbf{C} \in \mathcal{K}$.

Recall that a Γ_L -structure is *(ultra)homogeneous* if it is countable (possibly infinite) and every isomorphism between finite substructures extends to an automorphism. A Γ_L -structure \mathbf{A} is *locally finite* if the \mathbf{A} -closure of every finite subset of A is finite. We focus on locally finite structures only. In this context, (locally finite) homogeneous structures are characterised by the properties of their finite substructures. We denote by $\text{Age}(\mathbf{A})$ the class of all finite structures which embed to \mathbf{A} . For a class \mathcal{K} of relational structures, we denote by $\text{Age}(\mathcal{K})$ the class $\bigcup_{\mathbf{A} \in \mathcal{K}} \text{Age}(\mathbf{A})$.

The following is one of the cornerstones of model theory.

Theorem 1.1.3 (Fraïssé [Fra86], see e. g. [Hod93]). *Let Γ_L be language and let \mathcal{K} be a class of finite Γ_L -structures with only countably many non-isomorphic structures.*

- (a) *A class \mathcal{K} is the age of a countable locally finite homogeneous structure \mathbf{H} if and only if \mathcal{K} is an amalgamation class.*
- (b) *If the conditions of (a) are satisfied then the structure \mathbf{H} is unique up to isomorphism.*

Generalising notions of a graph clique and a Gaifman graph clique, we use the following:

Definition 1.1.4 (Irreducible structure [EHN17]). A Γ_L -structure is *irreducible* if it is not the free amalgamation of its proper substructures.

The following captures the right notion of mapping which will be used to present general constructions in Section 1.5.

Definition 1.1.5 (Homomorphism-embedding [HN19]). Let \mathbf{A} and \mathbf{B} be Γ_L -structures. A homomorphism $f: \mathbf{A} \rightarrow \mathbf{B}$ is a *homomorphism-embedding* if the restriction $f|_{\mathbf{C}}$ is an embedding whenever \mathbf{C} is an irreducible substructure of \mathbf{A} .

Given a family \mathcal{F} of Γ_L -structures, we denote by $\text{Forb}_{\text{he}}(\mathcal{F})$ the class of all finite or finite Γ_L -structures \mathbf{A} . such that there is no $\mathbf{F} \in \mathcal{F}$ with a homomorphism-embedding $\mathbf{F} \rightarrow \mathbf{A}$.

1.2 EXTENSION PROPERTY FOR PARTIAL AUTOMORPHISMS
(EPPA)

A *partial automorphism* of a Γ_L -structure \mathbf{A} is an isomorphism $f: \mathbf{C} \rightarrow \mathbf{C}'$ where \mathbf{C} and \mathbf{C}' are substructures of \mathbf{A} (note that this also includes a full permutation $f_L \in \Gamma_L$ of the language).

Definition 1.2.1 (EPPA [Hru92]). Let Γ_L be a language equipped with a permutation group. We say that a class \mathcal{K} of finite Γ_L -structures has the *extension property for partial automorphisms* (EPPA, sometimes called the *Hrushovski property*) if for every $\mathbf{A} \in \mathcal{K}$ there is $\mathbf{B} \in \mathcal{K}$ such that \mathbf{A} is a substructure of \mathbf{B} and every partial automorphism of \mathbf{A} extends to an automorphism of \mathbf{B} .

We call \mathbf{B} satisfying the condition of Definition 1.2.1 an *EPPA-witness* of \mathbf{A} . \mathbf{B} is *irreducible structure faithful* (with respect to \mathbf{A}) if it has the property that for every irreducible substructure \mathbf{C} of \mathbf{B} there exists an automorphism g of \mathbf{B} such that $g(C) \subseteq A$. The notion of faithful EPPA-witnesses was introduced by Hodkinson and Otto [HO03].

Observation 1.2.2. *Every hereditary isomorphism-closed class of finite Γ_L -structures which has EPPA and the joint embedding property (see Definition 1.1.2) is an amalgamation class.*

Proof. Let \mathcal{K} be such a class and let $\mathbf{A}, \mathbf{B}_1, \mathbf{B}_2 \in \mathcal{K}$, $\alpha_1: \mathbf{A} \rightarrow \mathbf{B}_1$, $\alpha_2: \mathbf{A} \rightarrow \mathbf{B}_2$ be as in Definition 1.1.2. Let \mathbf{B} be the joint embedding of \mathbf{B}_1 and \mathbf{B}_2 (that is, we have embeddings $\beta'_1: \mathbf{B}_1 \rightarrow \mathbf{B}$ and $\beta'_2: \mathbf{B}_2 \rightarrow \mathbf{B}$) and let \mathbf{C} be an EPPA-witness for \mathbf{B} .

Let φ be a partial automorphism of \mathbf{B} sending $\alpha_1(\mathbf{A})$ to $\alpha_2(\mathbf{A})$ and let θ be its extension to an automorphism of \mathbf{C} . Finally, put $\beta_1 = \theta \circ \beta'_1$ and $\beta_2 = \beta'_2$. It is easy to check that β_1 and β_2 certify that \mathbf{C} is an amalgamation of \mathbf{B}_1 and \mathbf{B}_2 over \mathbf{A} with respect to α_1 and α_2 . \square

EPPA was introduced by Hodges, Hodkinson, Lascar, and Shelah [HHLs93] to show the existence of ample generics (see e. g. [Sin17b]). For this application an additional property of the class is needed.

Definition 1.2.3 (APA [HHLs93]). Let \mathcal{K} be a class of finite Γ_L -structures. We say that \mathcal{K} has the *amalgamation property with automorphisms* (APA) if for every $\mathbf{A}, \mathbf{B}_1, \mathbf{B}_2 \in \mathcal{K}$ such that $\mathbf{A} \subseteq \mathbf{B}_1, \mathbf{B}_2$ there exists $\mathbf{C} \in \mathcal{K}$ which is an amalgamation of \mathbf{B}_1 and \mathbf{B}_2 over \mathbf{A} , has $\mathbf{B}_1, \mathbf{B}_2 \subseteq \mathbf{C}$ and furthermore the following holds:

For every pair of automorphisms $f_1 \in \text{Aut}(\mathbf{B}_1)$, $f_2 \in \text{Aut}(\mathbf{B}_2)$ such that $f_i(A) = A$ for $i \in \{1, 2\}$ and $f_1|_A = f_2|_A$, there is an automorphism $g \in \text{Aut}(\mathbf{C})$ such that $g|_{B_i} = f_i$ for $i \in \{1, 2\}$.

Siniora and Solecki [Solo9, SS19] strengthened the notion of EPPA in order to get a dense locally finite subgroup of the automorphism group of the corresponding Fraïssé limit.

Definition 1.2.4 (Coherent maps [SS19]). Let X be a set and \mathcal{P} be a family of partial bijections between subsets of X . A triple (f, g, h) from \mathcal{P} is called a *coherent triple* if

$$\text{Dom}(f) = \text{Dom}(h), \text{Range}(f) = \text{Dom}(g), \text{Range}(g) = \text{Range}(h)$$

and

$$h = g \circ f.$$

Let X and Y be sets, and \mathcal{P} and \mathcal{Q} be families of partial bijections between subsets of X and between subsets of Y , respectively. A function $\varphi: \mathcal{P} \rightarrow \mathcal{Q}$ is said to be a *coherent map* if for each coherent triple (f, g, h) from \mathcal{P} , its image $(\varphi(f), \varphi(g), \varphi(h))$ in \mathcal{Q} is also coherent.

Definition 1.2.5 (Coherent EPPA [SS19]). A class \mathcal{K} of finite Γ_L -structures is said to have *coherent EPPA* if \mathcal{K} has EPPA and moreover the extension of partial automorphisms is coherent. That is, for every $\mathbf{A} \in \mathcal{K}$, there exists $\mathbf{B} \in \mathcal{K}$ such that $\mathbf{A} \subseteq \mathbf{B}$ and every partial automorphism f of \mathbf{A} extends to some $\hat{f} \in \text{Aut}(\mathbf{B})$ with the property that the map φ from the partial automorphisms of \mathbf{A} to $\text{Aut}(\mathbf{B})$ given by $\varphi(f) = \hat{f}$ is coherent. We also say that \mathbf{B} is a *coherent EPPA-witness* for \mathbf{A} .

1.3 RAMSEY CLASSES

For L -structures \mathbf{A}, \mathbf{B} , we denote by $\binom{\mathbf{B}}{\mathbf{A}}$ the set of all embeddings $\mathbf{A} \rightarrow \mathbf{B}$. Using this notation, the definition of a Ramsey class gets the following form:

Definition 1.3.1 (Ramsey class). Let L be a language. A class \mathcal{K} of finite L -structures is a *Ramsey class* (or has the *Ramsey property*) if for every two L -structures $\mathbf{A} \in \mathcal{K}$ and $\mathbf{B} \in \mathcal{K}$ and for every positive integer k there exists a L -structure $\mathbf{C} \in \mathcal{K}$ such that the following holds: For every partition of $\binom{\mathbf{C}}{\mathbf{A}}$ into k classes there is $\tilde{\mathbf{B}} \in \binom{\mathbf{C}}{\mathbf{B}}$ such that $\binom{\tilde{\mathbf{B}}}{\mathbf{A}}$ belongs to one class of the partition. It is usual to shorten the last part of the definition to $\mathbf{C} \rightarrow (\mathbf{B})_k^{\mathbf{A}}$.

Analogously to Observation 1.2.2 one can see the following:

Observation 1.3.2 (Nešetřil, 1980's [Neš89]). *Every hereditary isomorphism-closed Ramsey class of finite L -structures with the joint embedding property is an amalgamation class.*

Proof. Let \mathcal{K} be such a class and let $\mathbf{A}, \mathbf{B}_1, \mathbf{B}_2 \in \mathcal{K}$, $\alpha_1: \mathbf{A} \rightarrow \mathbf{B}_1$, $\alpha_2: \mathbf{A} \rightarrow \mathbf{B}_2$ be as in Definition 1.1.2. Let \mathbf{B} be the joint embedding of \mathbf{B}_1 and \mathbf{B}_2 and let $\mathbf{C} \rightarrow (\mathbf{B})_2^{\mathbf{A}}$. Colour the elements of $\binom{\mathbf{C}}{\mathbf{A}}$ according to whether or not they are contained in a copy of \mathbf{B}_1 in \mathbf{C} . Because \mathbf{C} is Ramsey there is a monochromatic copy $\tilde{\mathbf{B}} \in \binom{\mathbf{C}}{\mathbf{B}}$. As it contains a copy of \mathbf{B}_1 , all the copies of \mathbf{A} in it are in a copy of \mathbf{B}_1 in \mathbf{C} . But this includes the \mathbf{A} which is in the copy of \mathbf{B}_2 in $\tilde{\mathbf{B}}$. It follows that \mathbf{C} contains an amalgamation of \mathbf{B}_1 and \mathbf{B}_2 over \mathbf{A} . \square

1.4 NEŠETŘIL'S CLASSIFICATION PROGRAMME

The celebrated (Cherlin and Lachlan's) *classification programme of homogeneous structures* aims to provide full catalogues of countable homogeneous combinatorial structures of a given type. The most important cases where the classification is complete are:

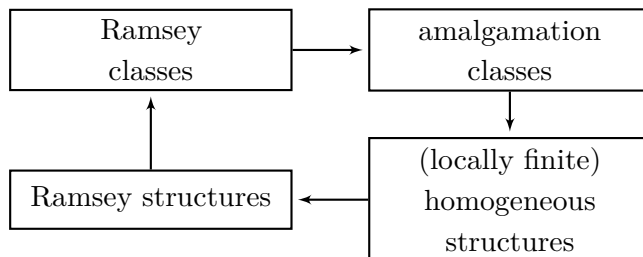
1. Schmerl's catalogue of all homogeneous partial orders [Sch79],
2. Lachlan and Woodrow's catalogue of all homogeneous simple graphs [LW80],
3. Lachlan's catalogue of all homogeneous tournaments [Lac84],
4. Cherlin's catalogue of all homogeneous digraphs [Che98] (this generalises catalogues 1 and 3),
5. Cherlin's catalogue of all ordered graphs [Che17], and
6. Braufeld's catalogue of homogeneous finite-dimensional permutation structures [Bra18b]. This was recently shown to be complete by Braufeld and Simon [Bra18a, BS18].

Several additional catalogues are known. An extensive list is given in the upcoming Cherlin's monograph [Che17]. Important for our discussion is also Cherlin's catalogue of metrically homogeneous graphs which is currently conjectured to be complete [Che17].

All these results give us a rich and systematic source of homogeneous structures.

1.4.1 *Classification of Ramsey expansions*

Shortly after observing the link between amalgamation classes, homogeneous structures and Ramsey classes, Nešetřil [Neš89] studied the catalogue of homogeneous graphs and verified that the corresponding Ramsey results are already known. In 2005, he proposed a project to analyse these catalogues and initiated the classification programme of Ramsey classes (which we refer to as *Nešetřil's classification programme*) [Neš05, HN05]. The overall idea is symbolised in [Neš05] as follows:



The individual arrows in the diagram can be understood as follows.

1. *Ramsey classes* \longrightarrow *amalgamation classes*. By Observation 1.3.2, every hereditary isomorphism-closed Ramsey class of finite L -structures with the joint embedding property is an amalgamation class.

The assumptions of Observation 1.3.2 are natural in the Ramsey context and it makes sense to restrict our attention only to the classes which satisfy them.

2. *Amalgamation classes* \longrightarrow *homogeneous structures*. By the Fraïssé theorem (Theorem 1.5.1), every amalgamation class of L -structures with countably many mutually non-isomorphic structures has a Fraïssé limit which is a (locally finite) homogeneous structure.

The additional assumption about the amalgamation class having only countably many mutually non-isomorphic structures is a relatively mild one. However, there are interesting and natural Ramsey classes not satisfying it, such as the class of all finite ordered metric spaces (see Section 1.6.3).

3. *Ramsey structures* \longrightarrow *Ramsey classes*. We call a structure *Ramsey* if it is locally finite and its age is a Ramsey class and thus this connection follows by the definition.

The difficult part of the diagram is thus the last arrow *homogeneous structures* \longrightarrow *Ramsey structures*. This motivates a question:

Is every locally finite homogeneous structure also a Ramsey structure?

The answer is most definitely negative, for reasons to be discussed below. For this reason, the precise formulation of this classification programme requires close attention.

It is a known fact that the automorphism group of every Ramsey structure fixes a linear order on vertices. This follows at once from the connection with topological dynamics [KPT05]; see [Bod15, Proposition 2.22] for a combinatorial proof. This is a very strong hypothesis—such a linear order is unlikely to appear in practice unless we began with a class of ordered structures. So it often occurs that the age \mathcal{K} of a homogeneous structure \mathbf{H} is not Ramsey. In many such cases, it can be “expanded” to a Ramsey class \mathcal{K}^+ by adding an additional binary relation \leq to the language and considering every structure in \mathcal{K} with every possible order. It is easy to check that \mathcal{K}^+ is an amalgamation class and the Fraïssé limit of \mathcal{K}^+ thus exists and leads to a Ramsey structure \mathbf{H}^+ . This Ramsey structure can be thought of as \mathbf{H}^+ with an additional “free” or “generic” linear order of vertices.

Example 1.4.1 (Countable random graph). Consider the class \mathcal{G} of all finite graphs and its Fraïssé limit \mathbf{R} . (\mathbf{R} is known as the *countable random graph*, or the *Rado graph*, and is one of the structures in the Lachlan and Woodrow's catalogue.) Because an order is not fixed by $\text{Aut}(\mathbf{R})$, we can consider the class \mathcal{G}^+ of all finite graphs “endowed” with a linear order of vertices. More formally, \mathcal{G}^+ is the class of all finite L -structures \mathbf{A} , where L consists of two binary relational symbols E (for edges) and \leq (for the order), $E_{\mathbf{A}}$ is a symmetric and irreflexive relation on A (representing the edges of a graph on A) and $\leq_{\mathbf{A}}$ is a linear order of A . By the Nešetřil–Rödl theorem (Theorem 1.6.1), \mathcal{G}^+ is Ramsey and thus its Fraïssé limit \mathbf{R}^+ is a Ramsey structure. In this sense, we completed the last arrow of Nešetřil's diagram.

Initially, the classification problem was understood in terms of adding linear orders freely when they were absent, and then confining one's attention to classes with the linear order present. This point of view is still implicit in [KPT05].

However, it turns out that it is necessary to consider more general expansions of languages than can be afforded using a single linear order. Furthermore, the topological dynamical view of [KPT05] clarifies what kind of Ramsey expansion we actually should look for, and this can be expressed very concretely in combinatorial terms.

On the other hand, if one allows more general expansions of the structure—for example, naming every point—then the Ramsey property may be obtained in a vacuous manner. This means that a conceptual issue of identifying the “correct” Ramsey expansions needs to be resolved first before returning to the technical issues of proving the existence of such an expansion, and constructing it explicitly. Nowadays, the conceptual issue may be considered to be satisfactorily resolved (at least provisionally) and progress on the resulting combinatorial problems is one of the main subjects of this habilitation.

We review the resolution of this issue and justify the following: the last arrow in Nešetřil's diagram represents the search for an *optimal expansion* of the homogeneous structure to a larger language, in a precise sense, and raises the question of the existence of such expansion.

Before entering into the technicalities associated with the conceptual issues, we present a critical example in which the Ramsey property requires something more than the addition of a linear order, and we describe the optimal solution according to the modern point of view.

Example 1.4.2 (Generic local order). An early example (given by Laflamme, Nguyen Van Thé and Sauer [LNVTs10]) of a structure with a non-trivial optimal Ramsey expansion is the *generic local order*. This is a homogeneous tournament $\mathbf{S}(2)$ (which we view as an L -structure

where L consists of binary relational symbol E) defined as follows. Let \mathbb{T} denote the unit circle in the complex plane. Define an oriented graph structure on \mathbb{T} by declaring that there is an arc from x to y iff $0 < \arg(y/x) < \pi$. Call $\vec{\mathbb{T}}$ the resulting oriented graph. The dense local order is then the substructure $\mathbf{S}(2)$ of $\vec{\mathbb{T}}$ whose vertices are those points of \mathbb{T} with rational argument.

Another construction of $\mathbf{S}(2)$ is to start from the order of rationals (seen as a countable transitive tournament), randomly colour vertices with two colours and then reverse the direction of all arrows between vertices of different colours. In fact, this colouring is precisely the necessary Ramsey expansion. We thus consider the class of finite L^+ -structures where L^+ consists of a binary relation E (representing the directed edges) and a unary relation R (representing one of the colour classes). The linear ordering of vertices is implicit, but can be defined based on the relations E and R , putting $a \leq b$ if and only if either there is an edge from a to b and they belong to the same colour class, or there is an edge from b to a and they belong to different colour classes. The Ramsey property then follows by a relatively easy application of the Ramsey theorem.

The general notion of expansion (or, more particularly, *relational expansion*) that we work with comes from model theory.

Definition 1.4.3 (Expansion and reduct). Let $L^+ = L_{\mathcal{R}}^+ \cup L_{\mathcal{F}}^+$ be a language containing language $L = L_{\mathcal{R}} \cup L_{\mathcal{F}}$, extending it by relational symbols. By this we mean $L_{\mathcal{R}} \subseteq L_{\mathcal{R}}^+$ and $L_{\mathcal{F}} = L_{\mathcal{F}}^+$ and the arities of the relations and functions which belong to both L and L^+ are the same. For every structure $\mathbf{X} \in \text{Str}(L^+)$, there is a unique structure $\mathbf{A} \in \text{Str}(L)$ satisfying $A = X$, $R_{\mathbf{A}} = R_{\mathbf{X}}$ for every $R \in L_{\mathcal{R}}$ and $F_{\mathbf{A}} = F_{\mathbf{X}}$ for every $F \in L_{\mathcal{F}}$. We call \mathbf{X} a (*relational*) *expansion* (or a *lift*) of \mathbf{A} and \mathbf{A} is called the *L-reduct* (or the *L-shadow*) of \mathbf{X} .

Given languages L and L^+ and a class \mathcal{K} of L -structures, a class \mathcal{K}^+ of L^+ -structures is an *expansion* of \mathcal{K} if \mathcal{K} is precisely the class of all L -reducts of structures in \mathcal{K}^+ .

As mentioned earlier, every L -structure \mathbf{H} can be turned to a Ramsey structure by naming every point. This can be done by expanding the language L by infinitely many unary relational symbols and putting every vertex of \mathbf{H} to a unique relation. The Ramsey property then holds, since there are no non-trivial embeddings between structures from the age of \mathbf{H} . Clearly, additional restrictions on the expansions need to be made.

We now formulate a notion of “canonical” or “minimal” Ramsey expansion. We will give this first in purely combinatorial terms. In those terms, we seek a “precompact Ramsey expansion with the expansion

property” as defined below. But to understand why this is canonical (i. e., natural and unique up to bi-definability), we need to invoke notions and non-trivial results of topological dynamics.

Definition 1.4.4 (Precompact expansion [NVT13]). Let \mathcal{K}^+ be an expansion of a class of structures \mathcal{K} . We say that \mathcal{K}^+ is a *precompact expansion* of \mathcal{K} if for every structure $\mathbf{A} \in \mathcal{K}$ there are only finitely many structures $\mathbf{A}^+ \in \mathcal{K}^+$ such that \mathbf{A}^+ is an expansion of \mathbf{A} .

Definition 1.4.5 (Expansion property [NVT13]). Let \mathcal{K}^+ be an expansion of \mathcal{K} . For $\mathbf{A}, \mathbf{B} \in \mathcal{K}$ we say that \mathbf{B} has the *expansion property* for \mathbf{A} if for every expansion $\mathbf{B}^+ \in \mathcal{K}^+$ of \mathbf{B} and for every expansion $\mathbf{A}^+ \in \mathcal{K}^+$ of \mathbf{A} there is an embedding $\mathbf{A}^+ \rightarrow \mathbf{B}^+$.

\mathcal{K}^+ has the *expansion property* relative to \mathcal{K} if for every $\mathbf{A} \in \mathcal{K}$ there is $\mathbf{B} \in \mathcal{K}$ with the expansion property for \mathbf{A} .

Intuitively, precompactness means that the expansion is not very rich and the expansion property then shows that it is minimal possible. To further motivate these concepts, we review the key connections to topological dynamics.

We consider the automorphisms group $\text{Aut}(\mathbf{H})$ as a Polish topological group by giving it the topology of pointwise convergence. Recall that a topological group Γ is *extremely amenable* if whenever X is a Γ -flow (that is, a non-empty compact Hausdorff Γ -space on which Γ acts continuously), then there is a Γ -fixed point in X . See [NVT15] for details.

In 1998, Pestov [Pes98] used the classical Ramsey theorem to show that the automorphism group of the order of rationals is extremely amenable. Two years later, Glasner and Weiss [GW02] proved (again applying the Ramsey theorem) that the space of all linear orderings on a countable set is the universal minimal flow of the infinite permutation group. In 2005, Kechris, Pestov and Todorćević introduced the general framework (which we refer to as *KPT-correspondence*) connecting Fraïssé theory, Ramsey classes, extremely amenable groups and metrizable minimal flows. Subsequently, this framework was generalised to the notion of Ramsey expansions [NVT13, NVT09, MNVTT15, Zuc16] with main results as follows:

Theorem 1.4.6 (Kechris, Pestov, Todorćević [KPT05, Theorem 4.8]). *Let \mathbf{H} be locally finite homogeneous L -structure. Then $\text{Aut}(\mathbf{H})$ is extremely amenable if and only if $\text{Age}(\mathbf{H})$ is a Ramsey class.*

Theorem 1.4.6 is often formulated with the additional assumption that $\text{Age}(\mathbf{H})$ is rigid (i.e. no structure in $\text{Age}(\mathbf{H})$ has non-trivial automorphisms). This is however implied by our definition of a Ramsey class (Definition 1.3.1) which colours embeddings. This definition implies

rigidity. As mentioned earlier, in addition to having a rigid age, the automorphism group of \mathbf{H} must also fix a linear order.

Recall that a Γ -flow is *minimal* if it admits no nontrivial closed Γ -invariant subset or, equivalently, if every orbit is dense. Among all minimal Γ -flows, there exists a canonical one, known as the *universal minimal Γ -flow*. Precompact Ramsey expansions with the expansion property relate to universal minimal flows as follows.

Theorem 1.4.7 (Melleray, Nguyen Van Thé, and Tsankov [MNVT15, Corollary 1.3]). *Let \mathbf{H} be a locally finite homogeneous structure and let $\Gamma = \text{Aut}(\mathbf{H})$. The following are equivalent:*

1. *The universal minimal flow of Γ is metrizable and has a comeagre orbit.*
2. *The structure \mathbf{H} admits a precompact expansion \mathbf{H}^+ whose age has the Ramsey property, and has the expansion property relative to $\text{Age}(\mathbf{H})$.*

Because the metrizable minimal flow is unique, a corollary of Theorem 1.4.7 is that for a given homogeneous structure \mathbf{H} there is, up to bi-definability, at most one Ramsey expansion \mathbf{H}^+ such that $\text{Age}(\mathbf{H}^+)$ is a precompact Ramsey expansion of $\text{Age}(\mathbf{H})$ with the expansion property (relative to \mathbf{H}). We will thus call such an expansion the *canonical Ramsey expansion*.

The classification programme of Ramsey classes thus turns into two questions. Given a locally finite homogeneous L -structure \mathbf{H} , we ask the following:

- Q1** Is there a Ramsey structure \mathbf{H}^+ which is a (relational) expansion of \mathbf{H} such that $\text{Age}(\mathbf{H}^+)$ is a precompact expansion of $\text{Age}(\mathbf{H})$? (Possibly with $\mathbf{H}^+ = \mathbf{H}$.)

If the answer to **Q1** is positive, we know that \mathbf{H}^+ can be chosen so that $\text{Age}(\mathbf{H}^+)$ has the expansion property relative to $\text{Age}(\mathbf{H})$ [KPT05, Theorem 10.7]. We can moreover ask.

- Q2** If the answer to **Q1** is positive, can we give an explicit description of \mathbf{H}^+ which additionally satisfies that the $\text{Age}(\mathbf{H}^+)$ has the expansion property with respect to $\text{Age}(\mathbf{H})$? In other words, can we describe the canonical Ramsey expansion of \mathbf{H} ?

In 2013, the classification programme in this form was first completed by Jasiński, Laflamme, Nguyen Van Thé and Woodrow [JLNV14] for the catalogue of homogeneous digraphs. More examples are discussed below.

Remark 1.4.8. In addition to the universal minimal flow (Theorem 1.4.7), by a counting argument given by Angel, Kechris and Lyons [AKL14], knowledge of an answer to question Q2 often gives amenability of $\text{Aut}(\mathbf{H})$ and under somewhat stronger assumptions also shows that $\text{Aut}(\mathbf{H})$ is uniquely ergodic. See [Sok15, PS17, Jah19] for an initial progress on the classification programme in this direction.

1.4.2 Classification of EPPA classes

In the light of recent connections between EPPA and Ramsey classes, we can extend this programme to also provide catalogues of classes with EPPA. The basic question for a given a locally finite homogeneous Γ_L -structure \mathbf{H} is simply:

Q3 Does the class $\text{Age}(\mathbf{H})$ have EPPA?

This is an interesting question from the combinatorial point of view and again has number of applications. Motivated by a group-theoretical context one can additionally consider the following questions:

Q4 If the answer to Q3 is positive, does it have coherent EPPA (Definition 1.2.5)?

Q5 If the answer to Q3 is positive, does it have APA (Definition 1.2.3)?

In this sense, the classification was first considered by Aranda, Bradley-Williams, Hubička, Karamanlis, Kompatscher, Konečný and Pawliuk for metrically homogeneous graphs [ABWH⁺17b].

Remark 1.4.9 (On group-theoretical context). By a result of Kechris and Rosendal [KR07], a positive answer to Q3 implies amenability of $\text{Aut}(\mathbf{H})$. This is a sufficient but not a necessary condition. For example, the automorphism group of the order of rationals is amenable (because it is extremely amenable) but does not have EPPA.

A positive answer to Q4 implies the existence of a dense locally finite subgroup [Solog, SS19]. The first known example, where an answer to question Q4 seems to be non-trivial (and presently open) is the class of two-graphs where EPPA was shown by Evans, Hubička, Konečný and Nešetřil [EHKN19]. The existence of a dense locally finite subgroup here however follows by a different argument.

1.4.3 Known classifications

The Ramsey part of the classification programme was completed for the following catalogues:

1. The catalogue of homogeneous graphs (Nešetřil [Neš89]). We remark that by today's view of the programme, the original 1989 results are incomplete because only expansions by free orderings are considered. The remaining cases (of disjoint unions of cliques and their complements) are however very simple.
2. The catalogue of directed graphs (Jasiński, Laflamme, Nguyen Van Thé and Woodrow [JLNVTW14]).
3. Completing the first two catalogues also covers all cases of the catalogue of homogeneous ordered graphs with the exception of the class of all finite partial orders with a linear extension which is known to be Ramsey, too (Section 1.6.2).
4. The conjectured-to-be-complete catalogue of metrically homogeneous graphs (Aranda, Bradley-Williams, Hubička, Karamanlis, Kompatscher, Konečný, Pawliuk [ABWH⁺17b, Theorem 1.1]).

Even though this point has apparently not been explicitly stated before, the EPPA classification is finished for the catalogue of homogeneous graphs. In fact, this catalogue consists of only 5 types of structures and their complements:

1. The countable random graph \mathbf{R} . $\text{Age}(\mathbf{R})$ is the class of all finite graphs for which EPPA was shown by Hrushovski [Hru92].
Coherent EPPA was given by Siniora and Solecki [SS19] and APA is trivial for every free amalgamation class.
2. Generic K_k -free graphs \mathbf{R}_k , $k \geq 3$. $\text{Age}(\mathbf{R}_k)$ is the class of all K_k -free graphs. For $k = 3$, EPPA was shown by Herwig [Her95] and he later generalised the construction to $k > 3$ [Her98].
Again, the existence of coherent EPPA extensions (using a construction by Hodkinson and Otto [HO03]) was verified by Siniora and Solecki [SS19] and APA is trivial.
3. The class of all graphs consisting of at most n cliques each of size at most k for a given $n, k \in \mathbb{N} \cup \{\infty\}$, $n + k = \infty$. Proving (coherent) EPPA for the ages of these structures is an easy exercise (building on the fact that every partial permutation extends coherently to a permutation).

The class has APA if and only if $n, k \in \{1, \infty\}$. If $n \geq 2$ is finite, one may consider an amalgamation of an n -anti-clique (that is, a graph with n vertices and no edges) and a 2-anti-clique over the empty graph. Similarly, for $k \geq 2$ finite, one can consider an amalgamation of a k -clique and a 2-clique over a vertex. These amalgamations are counter-examples to APA.

In this situation, it is possible to modify the structures by adding vertices representing imaginaries (Section 1.6.6) and functions representing the algebraic closures (Section 1.6.5), including, possibly, constants for the algebraic closure of the empty set. The age of the resulting structure then has coherent EPPA and APA.

It is also clear that if \mathcal{K} is a class of graphs with EPPA, then the class of all complements of graphs in \mathcal{K} has EPPA, too.

EPPA for ages of homogeneous directed graphs was analysed by Pawliuk and Sokić [PS17]. Their work leaves several open cases: n -partite tournaments, semi-generic tournaments (for both these cases EPPA was claimed recently by Hubička, Jahel, Konečný, and Sabok [HJKS19b]), tournaments (which present a well known open problem in the area asked in 2000 by Herwig and Lascar [HL00]), directed graphs omitting an independent set of size $n \geq 2$ and 2-covers of generic tournaments. The last two cases appear to be very similar to the case of tournaments.

Finally, Aranda, Bradley-Williams, Hubička, Karamanlis, Kompatscher, Konečný and Pawliuk studied EPPA for ages of metric spaces associated with metrically homogeneous graphs (and, for the first time, considered both the Ramsey property and EPPA together) and characterised all structures in the catalogue with the exception of special cases of antipodal metrically homogeneous graphs [ABWH⁺17b, Theorem 1.2]. For the Ramsey property (and partly also for EPPA), the analysis of metrically homogeneous graphs was, for the first time, done using the general results which are introduced in Section 1.5. That work can also be seen as a confirmation of the effectivity of these methods. EPPA for the remaining case (of antipodal metric spaces) was recently proved by Konečný [Kon19b] generalising a result for diameter three by Evans, Hubička, Konečný and Nešetřil [EHKN19].

Results on metrically homogeneous graphs are all based on a close study of *completion algorithms* for partial structures. The completion algorithm introduced then led to a new line of research on generalised metric spaces [HKN17, HKN18, Kon18, Kon19a]. This in turn led to new tools for proving simplicity of the automorphism groups of a wide range of structures [EHKL19], developing a method of Tent and Ziegler with its roots in model theory (stability theory) [TZ13b, TZ13a, Li18]. As a result, there is a new framework for understanding various structural properties of these combinatorial structures [HKK18] which may even lead to a more profound understanding of their classification.

1.5 GENERAL CONSTRUCTIONS

In this section, we present general theorems which are then used to show that a given class of structures has EPPA or the Ramsey property. These results unify essentially all earlier results in the area under a common framework as discussed in Section 1.6 which also lists all known examples which were not yet covered by this work.

1.5.1 *Unrestricted theorems*

The starting points for subsequent constructions of EPPA-witnesses and Ramsey structures are the following two theorems. We will refer to them as *unrestricted theorems*, because no restrictions on the constructed structures are given.

Theorem 1.5.1 (Hubička, Konečný, Nešetřil 2019 [HKN19b, Theorem 1.3]). *Let Γ_L be a finite language equipped with a permutation group where all function symbols are unary (no restrictions are given on the relational symbols) and \mathbf{A} be a finite Γ_L -structure. Then there exists a finite coherent EPPA-witness \mathbf{B} for \mathbf{A} .*

Consequently, the class $\text{Str}(\Gamma_L)$ of all finite Γ_L -structures has coherent EPPA.

For a language L containing a binary relation \leq , we say that an L -structure \mathbf{A} is *ordered* if $\leq_{\mathbf{A}}$ is a linear order of A .

Theorem 1.5.2 (Hubička, Nešetřil 2019 [HN19, Theorem 2.19]). *Let L be a language containing a binary relation \leq and \mathbf{A}, \mathbf{B} be finite ordered L -structures. Then there exists a finite ordered L -structure \mathbf{C} such that $\mathbf{C} \rightarrow (\mathbf{B})_2^{\mathbf{A}}$.*

Consequently, the class $\overrightarrow{\text{Str}}(L)$ of all finite ordered L -structures is Ramsey.

Recall that (under mild assumptions) every Ramsey class fixes a linear order (Section 1.4). For this reason, the assumption on structures being ordered in Theorem 1.5.2 can not be omitted and thus Theorem 1.5.2 is the most general unrestricted Ramsey theorem for finite L -structures.

Except for degenerated examples, EPPA classes never consists of ordered structures: automorphism groups of finite linearly ordered chains are trivial and thus can not extend any non-trivial partial automorphism (such as one sending a vertex to another). Classes with EPPA and Ramsey property are thus basically disjoint. Yet the similarity between both theorems shows that both types of classes are related.

Despite the compact formulations, both theorems are a result of a long development.

Remark 1.5.3 (History of unrestricted EPPA results). Theorem 1.5.1 is a generalisation of the original 1992 result of Hrushovski [Hru92] for graphs and Herwig’s strengthening to relational structures [Her95]. A variant for structures with relations and unary function was shown by Evans, Hubička, Nešetřil [EHN17], aiming to solve problems arising from the study of sparse graphs [EHN19]. The notion of Γ_L -structures is motivated by a lemma on permomorphisms used by Herwig [Her98, Lemma 1]. It was noticed by Ivanov [Iva15] that this lemma is of an independent interest and can be used to show EPPA for classes with definable equivalences.

There are multiple strategies for prove Theorem 1.5.1 for relational structures, including group-theoretical ones (sometimes referred to as Mackey’s constructions) [Hru92, HL00, Sab17] and an easy combinatorial construction based on intersection graphs [HL00]. The approach taken in our proof is new, inspired by a related result of Hodkinson and Otto [HO03]. We refer to it as a *valuation construction*.

Remark 1.5.4 (History of unrestricted structural Ramsey results). Generalising earlier results for colouring vertices [Fol70, NR76a, NR77] and edges of graphs, Theorem 1.5.2 for relational structures was proved by Nešetřil and Rödl in 1977 [NR77] and, independently, by Abramson and Harrington in 1978 [AH78]. A strengthening of this theorem for classes of structures with relations and unary functions is relatively easy to prove and was done in a special case by Hubička and Nešetřil [HN18a] (for bowtie-free graphs) and independently by Sokić [Sok16] (for structures with unary functions only). The proof strategies used in both of these papers turned out to be unnecessarily complex. For both Ramsey and EPPA, unary relations and functions can be added by an incremental construction on top of an existing Ramsey structure in a relational language. This general phenomenon is discussed by Hubička and Nešetřil [HN19, Section 4.3.1].

The final, and substantial, strengthening was to introduce a construction for structures in languages with function symbols of higher arities. It is interesting to note that while there are multiple proofs of the unrestricted theorem for structures in relational languages [NR89, Prö13, Sau06], the only known proof of Theorem 1.5.2 uses a recursive variant of the *partite construction*—presently the most versatile method of constructing Ramsey objects developed by Nešetřil and Rödl in a series of papers since late 1970’s [NR76b, NR77, NR79, NR81, NR82, NR83, NR84, NR87, NR89, NR90, Neš07]. The recursive variant of the partite construction was introduced by Hubička and Nešetřil to prove Theorem 1.5.2. A special case was independently used by Bhat, Nešetřil, Reiher and Rödl to obtain the Ramsey property of the class of all finite ordered partial Steiner systems [BNRR18].

Remark 1.5.5 (Infinitary structural Ramsey theorems). An infinitary variant of the unrestricted structural Ramsey theorem for graphs is shown by Sauer [Sau06] (generalising work of Devlin [Dev79]). This is a highly non-trivial strengthening of the finitary version with additional consequences for the automorphism groups [Zuc17]. A generalisation for structures in a finite language with relations and unary functions was recently claimed by Balko, Chodounský, Hubička, Konečný and Vena [BCH⁺19]. The first restricted theorems in this area were given recently by Dobrinen [Dob18, Dob19]. Her proofs combine many techniques and are very technically challenging.

1.5.2 Sparsening constructions

In order to work with classes of structures satisfying additional axioms (such as metric spaces or triangle free graphs), it is usual to first apply the unrestricted theorems (Theorems 1.5.1 and 1.5.2) and then use the resulting structures as a template to build bigger, and more sparse, structures with the desired local properties. This is a nature of several earlier proofs of EPPA and the Ramsey property [Neš07, DR12, HL00, HO03, Con19] and can be more systematically captured by the following definition and theorems:

Definition 1.5.6 (Tree amalgamation [HKN19b, Definition 7.1]). Let Γ_L be a language equipped with a permutation group and let \mathbf{A} be a finite irreducible Γ_L -structure (Definition 1.1.4). We inductively define what a *tree amalgamation of copies of \mathbf{A}* is.

1. If \mathbf{D} is isomorphic to \mathbf{A} then \mathbf{D} is a tree amalgamation of copies of \mathbf{A} .
2. If \mathbf{B}_1 and \mathbf{B}_2 are tree amalgamations of copies of \mathbf{A} and \mathbf{D} is a Γ_L -structure with an embedding to all of \mathbf{A} , \mathbf{B}_1 and \mathbf{B}_2 , then the free amalgamation of \mathbf{B}_1 and \mathbf{B}_2 over \mathbf{D} is also a tree amalgamation of copies of \mathbf{A} .

Recall the definition of homomorphism-embedding (Definition 1.1.3). In these terms, the sparsening constructions can be stated as follows.

Theorem 1.5.7 (Hubička, Konečný, Nešetřil 2019). *Let Γ_L be a finite language equipped with a permutation group where all function symbols are unary, $n \geq 1$, \mathbf{A} a finite irreducible Γ_L -structure and \mathbf{B}_0 its finite EPPA-witness. Then there exists a finite EPPA-witness \mathbf{B} for \mathbf{A} such that*

1. *there is a homomorphism-embedding (a projection) $\mathbf{B} \rightarrow \mathbf{B}_0$,*

2. for every substructure \mathbf{B}' of \mathbf{B} with at most n vertices there exists a structure \mathbf{T} which is a tree amalgamation of copies of \mathbf{A} and a homomorphism-embedding $\mathbf{B}' \rightarrow \mathbf{T}$,
3. \mathbf{B} is irreducible structure faithful.

If \mathbf{B}_0 is coherent, then \mathbf{B} is coherent, too.

Theorem 1.5.8 (Hubička, Nešetřil 2019). *Let L be a language, $n \geq 1$, \mathbf{A}, \mathbf{B} finite irreducible L -structures and \mathbf{C}_0 a finite L -structure such that $\mathbf{C}_0 \rightarrow (\mathbf{B})_2^{\mathbf{A}}$. Then there exists a finite L -structure \mathbf{C} such that $\mathbf{C} \rightarrow (\mathbf{B})_2^{\mathbf{A}}$ and*

1. there exists a homomorphism-embedding $\mathbf{C} \rightarrow \mathbf{C}_0$,
2. for every substructure \mathbf{C}' of \mathbf{C} with at most n vertices there exists a structure \mathbf{T} which is tree amalgamation of copies of \mathbf{A} and a homomorphism-embedding $\mathbf{C}' \rightarrow \mathbf{T}$,
3. every irreducible substructure of \mathbf{C} is also a substructure of a copy of \mathbf{B} in \mathbf{C} .

While not stated in this form, Theorem 1.5.7 follows by a proof of Lemma 2.8 in [HKN19b] and Theorem 1.5.8 is a direct consequence of the iterated partite construction as used in the proof of Lemmas 2.30 and 2.31 in [HN19]. While the underlying combinatorics for EPPA and Ramsey constructions are very different, the overall structure of the proofs is the same. To prove Theorem 1.5.7, one repeats the valuation construction n times, each time taking the result of the previous step as a template to build a new structure. Analogously, Theorem 1.5.8 is proved by repeating the partite construction n times with a similar setup.

1.5.3 Structural conditions on amalgamation classes

Theorems 1.5.7 and 1.5.8 can be used as “black boxes” to show EPPA and the Ramsey property for almost all known examples. However, to make their application easier, it is useful to introduce some additional notions.

Recall that by Observations 1.2.2 and 1.2.2, all sensible candidates (that is, hereditary isomorphism-closed classes of finite structures with the joint embedding property) for EPPA and the Ramsey property are amalgamation classes. Since neither EPPA or the Ramsey property is implied by the amalgamation property in full generality (as can be demonstrated by counter-examples), the aim of this section is to give structural conditions which are sufficient to prove EPPA or the Ramsey property for a given amalgamation class \mathcal{K} .

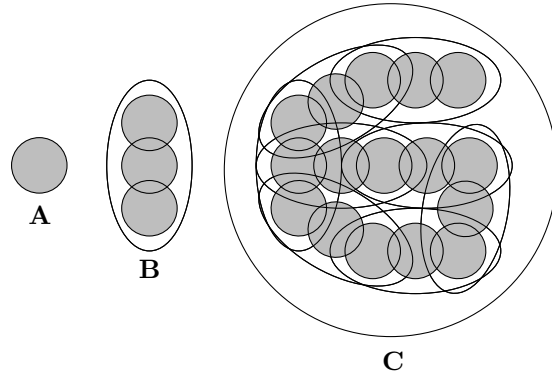


Figure 1.2: Construction of a Ramsey object by multiamalgamation.

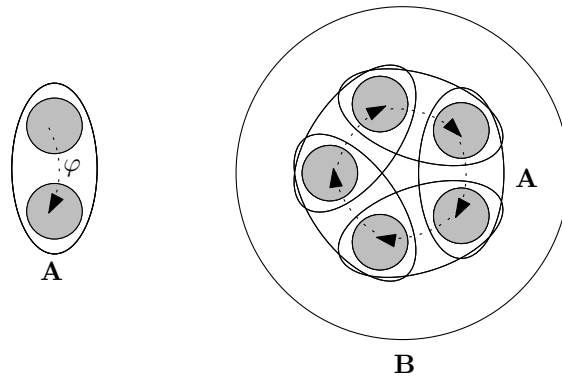


Figure 1.3: Extending one partial automorphism φ .

We will make a technical assumption that all structures in the class \mathcal{K} we are working with are irreducible. This will be helpful in formulating the conditions dealing with “structures with holes”. Irreducibility can be easily accomplished for any amalgamation class \mathcal{K} by considering its expansion adding a binary symbol R and putting for every $(u, v) \in R_{\mathbf{A}}$ for every $\mathbf{A} \in \mathcal{K}$ and every $u, v \in A$.

The following concepts were introduced by Hubička and Nešetřil in the Ramsey context [HN19] and subsequently adjusted for EPPA by Hubička, Konečný and Nešetřil [HKN19b]. We combine both approaches.

At an intuitive level, it is not hard to see that both Ramsey constructions and EPPA constructions can be seen as series of amalgamations all performed at once as schematically depicted in Figures 1.2 and 1.3. These amalgamations must “close cycles” and can not be just tree amalgamations in the sense of Definition 1.5.6.

Instead of working with complicated amalgamation diagrams, we split the process into two steps—the *construction* of the free multiamalgamation (which yields an incomplete, or “partial”, structure) followed by a *completion* (cf. Bitterlich and Otto [Bit19]).

Definition 1.5.9 (Completion [HN19, Definition 2.5], [HKN19b, Definition 8.2]). Let \mathbf{C} be a Γ_L -structure. An irreducible Γ_L -structure \mathbf{C}' is a *completion* of \mathbf{C} if there exists a homomorphism-embedding $\mathbf{C} \rightarrow \mathbf{C}'$. If there is a homomorphism-embedding $\mathbf{C} \rightarrow \mathbf{C}'$ which is injective, we call \mathbf{C}' a *strong completion*.

We also say that a strong completion is *automorphism-preserving* if for every $\alpha \in \text{Aut}(\mathbf{C})$ there is $\alpha' \in \text{Aut}(\mathbf{C}')$ such that $\alpha \subseteq \alpha'$ and moreover the map $\alpha \mapsto \alpha'$ is a group homomorphism $\text{Aut}(\mathbf{C}) \rightarrow \text{Aut}(\mathbf{C}')$.

Of particular interest will be the question whether there exists a completion in a given class \mathcal{K} of structures. In this case we speak about a *\mathcal{K} -completion*.

For classes of irreducible structures, (strong) completion may be seen as a generalisation of (strong) amalgamation: Let \mathcal{K} be such a class. The (strong) amalgamation property of \mathcal{K} can be equivalently formulated as follows: For $\mathbf{A}, \mathbf{B}_1, \mathbf{B}_2 \in \mathcal{K}$ and embeddings $\alpha_1: \mathbf{A} \rightarrow \mathbf{B}_1$ and $\alpha_2: \mathbf{A} \rightarrow \mathbf{B}_2$, there is a (strong) \mathcal{K} -completion of the free amalgamation of \mathbf{B}_1 and \mathbf{B}_2 over \mathbf{A} with respect to α_1 and α_2 .

Observe that the free amalgamation is not in \mathcal{K} unless the situation is trivial. Free amalgamation results in a reducible structure, as the pairs of vertices where one vertex belongs to $\mathbf{B}_1 \setminus \alpha_1(\mathbf{A})$ and the other to $\mathbf{B}_2 \setminus \alpha_2(\mathbf{A})$ are never both contained in a single tuple of any relation. Such pairs can be thought of as *holes* and a completion is then a process of filling in the holes to obtain irreducible structures while preserving all embeddings of irreducible structures.

The key structural condition can now be formulated as follows:

Definition 1.5.10 (Locally finite subclass [HKN19b, Definition 8.3], [HN19, Definition 2.8]). Let \mathcal{E} be a class of finite Γ_L -structures and \mathcal{K} a subclass of \mathcal{E} consisting of irreducible structures. We say that the class \mathcal{K} is a *locally finite subclass* of \mathcal{E} if for every $\mathbf{A} \in \mathcal{K}$ and every $\mathbf{B}_0 \in \mathcal{E}$ there is a finite integer $n = n(\mathbf{A}, \mathbf{B}_0)$ such that every Γ_L -structure \mathbf{B} has a completion $\mathbf{B}' \in \mathcal{K}$ provided that it satisfies the following:

1. every irreducible substructure of \mathbf{B} lies in a copy of \mathbf{A} ,
2. there is a homomorphism-embedding from \mathbf{B} to \mathbf{B}_0 , and,
3. every substructure of \mathbf{B} on at most n vertices has a completion in \mathcal{K} .

We say that \mathcal{K} is a *locally finite automorphism-preserving subclass* of \mathcal{E} if \mathbf{B}' can always be chosen to be strong and automorphism-preserving.

The following results are our main tools for obtaining EPPA and Ramsey results. (And are, in fact, corollaries of Theorems 1.5.7 and 1.5.8.)

Theorem 1.5.11 (Hubička, Konečný, Nešetřil [HKN19b]). *Let Γ_L be a finite language equipped with a permutation group where all function symbols are unary (no restrictions are given on the relational symbols) and let \mathcal{E} be a class of finite irreducible Γ_L -structures with EPPA. Let \mathcal{K} be a hereditary locally finite automorphism-preserving subclass of \mathcal{E} with the strong amalgamation property. Then \mathcal{K} has EPPA.*

Moreover, if EPPA-witnesses in \mathcal{E} can be chosen to be coherent then EPPA-witnesses in \mathcal{K} can be chosen to be coherent, too.

Theorem 1.5.12 (Hubička, Nešetřil [HN19]). *Let L be a language, let \mathcal{R} be a Ramsey class of irreducible finite L -structures and let \mathcal{K} be a hereditary locally finite subclass of \mathcal{R} with the strong amalgamation property. Then \mathcal{K} is a Ramsey class.*

Explicitly: For every pair of structures $\mathbf{A}, \mathbf{B} \in \mathcal{K}$ there exists a structure $\mathbf{C} \in \mathcal{K}$ such that

$$\mathbf{C} \longrightarrow (\mathbf{B})_2^{\mathbf{A}}.$$

For applications, it is important that in many cases the existence of \mathcal{K} -completions and strong \mathcal{K} -completions coincide. This can be formulated as follows.

Proposition 1.5.13 (Hubička, Nešetřil [HN19, Proposition 2.6]). *Let L be a language such that all function symbols in L have arity one (there is no restriction on relational symbols) and let \mathcal{K} be a hereditary class of finite irreducible L -structures with the strong amalgamation property. A finite L -structure \mathbf{A} has a \mathcal{K} -completion if and only if it has a strong \mathcal{K} -completion.*

With this proposition at hand, verification of the condition given by Definition 1.5.10 can be carried out for many amalgamation classes. Several examples are worked out in [HN19, Section 4] and followup papers (see, for example, [Bra17, HKN17, ABWH⁺17b, ABWH⁺17a, Kon18, Kon19a, Kon19b]).

Let us discuss two examples which demonstrate the techniques for verifying local finiteness and also a simple situation where this condition does not hold.

Example 1.5.14 (Metric spaces with distances 1, 2, 3 and 4 [HN19, Example 2.9]). Consider a language L containing binary relations R^1 , R^2 , R^3 and R^4 which we understand as distances. Let \mathcal{E} be the class of all irreducible finite structures where all four relations are symmetric, irreflexive and every pair of distinct vertices is in precisely one of relations R^1 , R^2 , R^3 , or R^4 (\mathcal{E} may be viewed a class of 4-edge-coloured complete graphs). Let \mathcal{K} be a subclass of \mathcal{E} consisting of those structures which satisfy the triangle inequality (i. e. \mathcal{K} is the class of finite metric spaces with distances 1, 2, 3, and 4).

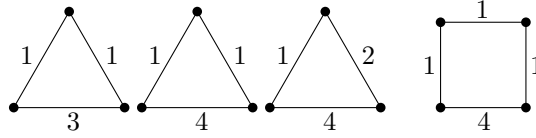


Figure 1.4: Cycles with no completion to a metric space with distances 1, 2, 3 and 4.

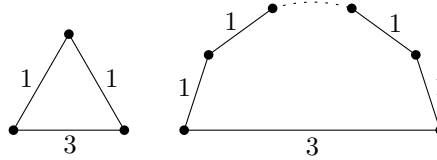


Figure 1.5: Cycles with no completion to a metric spaces with distances one and three.

It is not hard to verify that an L -structure \mathbf{B} which has a completion to some $\mathbf{B}_0 \in \mathcal{E}$ (meaning that all relations are symmetric and irreflexive and every pair of distinct vertices is in at most one relation) can be completed to a metric space if and only if it contains no non-metric triangles (i. e. triangles with distances 1-1-3, 1-1-4 or 1-2-4) and no 4-cycle with distances 1-1-1-4, see Figure 1.4. This can be done by computing the shortest distance among the edges present in the partial structure. Such completion is also clearly automorphism-preserving. It follows that \mathcal{K} is a locally finite subclass of \mathcal{E} and for every $\mathbf{C}_0 \in \mathcal{E}$ we can put $n(\mathbf{C}_0) = 4$.

Example 1.5.15 (Metric spaces with distances 1 and 3 [HN19, Example 2.10]). Now consider the class $\mathcal{K}_{1,3}$ of all metric spaces which use only distances one and three. It is easy to see that $\mathcal{K}_{1,3}$ is not a locally finite subclass of \mathcal{E} (as given in Example 1.5.14). To see that let $\mathbf{T} \in \mathcal{E}$ be the triangle with distances 1-1-3. Now consider a cycle \mathbf{C}_n of length n with one edge of distance three and the others of distance one (as depicted in Figure 1.5). \mathbf{T} is a completion of \mathbf{C}_n , however it has no $\mathcal{K}_{1,3}$ -completion. Moreover, every proper substructure of \mathbf{C}_n (that is, a path consisting of at most one edge of distance three and others of distance one) does have a $\mathcal{K}_{1,3}$ -completion. It follows that there is no $n(\mathbf{T})$ and thus $\mathcal{K}_{1,3}$ is not a locally finite subclass of \mathcal{E} .

We remark that an equivalent of Proposition 1.5.13 does not hold for languages with functions of greater arity. This can be demonstrated on a homogenization of the class of all finite graphs of girth 5 [HN19, Example 2.7].

1.6 KEY EXAMPLES

Thanks to the general constructions presented in Section 1.5 and systematic work on the classification programme, the landscape of known Ramsey and EPPA classes has recently changed significantly. Instead of several isolated examples, we nowadays know many classes which are too numerous to be fully covered by this introduction. We however list those which we consider most significant for developing the general theory.

1.6.1 Free amalgamation classes

The Nešetřil–Rödl theorem [NR77] was the first result which gave the Ramsey property of a class of structures satisfying additional axioms. Let us cite its 1989 formulation [NR89].

Theorem 1.6.1 (Nešetřil–Rödl theorem for hypergraphs [NR89]). *Let \mathbf{A} and \mathbf{B} be ordered hypergraphs, then there exists an ordered hypergraph \mathbf{C} such that $\mathbf{C} \rightarrow (\mathbf{B})_2^{\mathbf{A}}$.*

Moreover, if \mathbf{A} and \mathbf{B} do not contain an irreducible hypergraph \mathbf{F} (as a non-induced sub-hypergraph) then \mathbf{C} may be chosen with the same property.

In this formulation, it does not speak about an amalgamation class of structures, but it is not hard to work out that the original partite construction proof works in fact for all free amalgamation classes of relational structures expanded by a free linear order. This may be seen as an interesting coincidence, since the partite construction is not based only on a free amalgamation argument only. However, the connection is given by the following easy observation:

Observation 1.6.2. *For every free amalgamation class \mathcal{K} of Γ_L -structures there exists a family of irreducible structures \mathcal{F} such that \mathcal{K} is precisely the class of all finite structures \mathbf{A} for which there is no $\mathbf{F} \in \mathcal{F}$ which embeds into \mathbf{A} .*

Proof. Put \mathcal{F} to be all Γ_L -structures \mathbf{F} such that $\mathbf{F} \notin \mathcal{K}$ and every proper substructure of \mathbf{F} is in \mathcal{K} . It follows that all such structures \mathbf{F} are irreducible (otherwise one obtains a contradiction with \mathcal{K} being a free amalgamation class) and has the desired property. \square

This can be re-formulated as follows. Given a free amalgamation class \mathcal{K} and a structure $\mathbf{B} \in \mathcal{K}$, we know that every structure \mathbf{C} such that every irreducible substructure of \mathbf{C} is also a substructure of \mathbf{B} is in the class \mathcal{K} . This is precisely what the original formulation of the

Nešetřil–Rödl theorem gives. In the EPPA context, a stronger condition is accomplished by irreducible structure faithfulness.

It is thus well established that all free amalgamation classes of relational structures are Ramsey when expanded with a free linear order and this expansion has the expansion (ordering) property (see e. g. Nešetřil’s [Neš95] or Nguyen Van Thé’s [NVT15] surveys). All free amalgamation classes of relational structures also have (coherent) EPPA by an analogous result of Hodkinson and Otto [HO03], as observed by Siniora [Sin17b] (with coherence verified by Siniora and Solecki [SS19]).

Note that the class \mathcal{F} can also be seen as a class of minimal *obstacles*. The analysis of obstacles in a generalised sense of Definition 1.5.10 remains the main tool for obtaining Ramsey and EPPA results.

The situation is more complicated for languages involving functions. Forty years after the Nešetřil–Rödl theorem, Evans, Hubička and Nešetřil [EHN17, Theorem 1.3] showed that all free amalgamation classes are Ramsey when expanded by a free linear order. However, the expansion property does not necessarily hold [EHN17, Proposition 3.1]. The special (admissible) orderings which lead to a Ramsey class with the expansion property are explicitly described in Section 3.1 of [EHN17].

A coherent EPPA theorem for free amalgamation classes with unary functions was also shown by Evans, Hubička and Nešetřil [EHN17, Theorem 1.3]. A generalisation for languages with non-unary functions remains open.

Ramsey expansions of free amalgamation classes also follow by a direct application of Theorems 1.5.2 and 1.5.12 for $n = 1$. Similarly, a generalisation of [EHN17, Theorem 1.3] for Γ_L -structures in languages with unary functions follows by a direct application of Theorems 1.5.1 and 1.5.11 for $n = 1$. A special case of an EPPA construction for free amalgamation class with non-unary functions is discussed in [HKN19b, Section 9.3]. It makes a non-trivial use of the permutation group on the language, where the result of one EPPA construction defines the language and its permutation group for another EPPA construction. In general, however, the question of EPPA for free amalgamation classes in languages involving non-unary functions remains open.

1.6.2 *Partial orders*

The class of all partial orders expanded by a linear extension forms a Ramsey class. This result was announced by Nešetřil and Rödl in 1984 [NR84] and the first proof was published by Paoli, Trotter and Walker one year later [PTW85]. The original proof of Nešetřil and

Rödl using the partite construction was published only recently [NR18]. Several alternative proofs are known [Sok12, Maš18].

This is the first example where an interaction between the partial order in the class and a linear order of its expansion was observed (a phenomenon that generalises to other examples of reducts of partial [PPP⁺14] and total orders [JZ08]). It can be easily shown that partial orders do not form a Ramsey class when expanded by a free linear order [Sok12, Lemma 4].

The Ramsey property for partial orders with linear extensions can be shown by an easy application of Theorem 1.5.12 [HN19, Section 4.2.1]. In fact, this class served as a key motivation for the notion of locally finite subclass. As discussed in [HN19, Section 4.2.1], the choice of parameter n in Definition 1.5.10 depends on the number of vertices of \mathbf{B}_0 . In most other applications, the bound n can be chosen globally for all possible choices of \mathbf{B}_0 . This example demonstrates the power of the “ambient linear order” present in every Ramsey class.

Observe that the class of all partial order does not have EPPA for the same reason as in the case of total orders. This example thus shows the importance of the “base class” \mathcal{E} in the definition of locally finite subclass.

1.6.3 Metric spaces

The Ramsey property of linearly ordered metric spaces was shown by Nešetřil [Neš07]. He isolated the general form of the iterated partite construction which is also the basic mechanism of the proof of Theorem 1.5.8 (special cases of this technique have been used since late 1970’s [NR79], but here it appears in a very general form). A related construction was also independently used by Dellamonica and Rödl for the class of graphs with distance preserving embeddings [DR12].

The iterated partite construction has proven to be a useful tool for obtaining many additional results. However, quite surprisingly, a rather simple reduction to the (believed to be much easier) Ramsey property of the class of finite partial orders with a linear extension has recently been published by Mašulović [Maš18]. This construction does not however seem to extend to other cases on which the iterated partite construction method can be applied.

Metric spaces presented an important example in the study of EPPA classes, too. EPPA for metric spaces was obtained independently by Solecki [Solo5] and Vershik [Ver08]. Solecki’s proof is among the first applications of the deep Herwig–Lascaz theorem [HL00], while Vershik announced a direct proof which remains unpublished. Additional proofs were published by Pestov [Pes08], Rosendal [Ros11], Sabok [Sab17],

Theorem 8.3]. All these proofs are based on group-theoretical methods (the M. Hall theorem [Hal49], the Herwig–Lascar theorem [HL00, Ott17, SS19], the Ribes–Zalesskiĭ theorem [RZ93] or Mackey’s construction [Mac66]). A simple combinatorial proof was found by Hubička, Konečný and Nešetřil in 2018 [HKN19a], the core ideas of which were later developed to the form of Theorem 1.5.7.

Metric spaces with additional axioms are still presenting interesting challenges in the classification programme. Several special classes of metric spaces are considered by Nguyen Van Thé in his monograph [NVT10]. A generalisation of metric spaces was considered by Conant [Con19].

Analysis of local finiteness based on non-metric cycles is outlined in Example 1.5.14 and can be extended to various restricted cases of generalised and restricted metric spaces. This presents a currently active line of research [HN19, Section 4.2.2], [ABWH⁺17a, BCH⁺19, HKN17, HKN18, Kon18, Kon19a]. One of the important open questions in the area is the existence of a precompact Ramsey expansion (or EPPA) of the class of all finite affinely independent Euclidean metric spaces [NVT10].

1.6.4 *Homogenisations of classes defined by forbidden homomorphic images*

Fix a family \mathcal{F} of finite connected L -structures and consider the class $\mathcal{K}_{\mathcal{F}}$ of all countable structures \mathbf{A} such that there is no $\mathbf{F} \in \mathcal{F}$ with a monomorphism to \mathbf{A} . (For graphs this means that \mathbf{F} is isomorphic no subgraph \mathbf{S} of \mathbf{A} . Note that here \mathbf{S} is not necessarily induced.) In 1999, Cherlin, Shelah and Shi [CSS99] gave a structural condition for the existence of a *universal* structure $\mathbf{U} \in \mathcal{K}_{\mathcal{F}}$, that is a structure which contains a copy of every other structure in $\mathcal{K}_{\mathcal{F}}$.

Deciding about existence of a universal structure for a given class $\mathcal{K}_{\mathcal{F}}$ is a non-trivial task even for \mathcal{F} consisting of a single graph [CS01, CT07, CS07b, CS07a, Che11, CS13, CS]. However when \mathcal{F} is finite and closed for homomorphism-embedding images then a universal structure always exists. Hubička and Nešetřil [HN16, HN15] studied explicit constructions of universal structures for such classes $\mathcal{K}_{\mathcal{F}}$. (In earlier works, homomorphisms are used in place of homomorphism-embeddings.) This led to an explicit construction which expands class $\text{Forb}_{\text{he}}(\mathcal{F})$ to an amalgamation class by means of new relations (called a *homogenization* and first used in this context by Covington [Cov90]) and the universal structure is then the reduct of the Fraïssé limit of $\text{Forb}_{\text{he}}(\mathcal{F})$.

Work on a general theorem giving Ramsey expansions for $\text{Forb}_{\text{he}}(\mathcal{F})$ classes resulted in an unexpected generalisation in the form of Theorem 1.5.12. For that Proposition 1.5.13 giving close interaction between

the strong amalgamation and $\text{Forb}_{\text{he}}(\mathcal{F})$ classes was necessary. The relationship to homogenisations is exploited in [HN19, Section 3] and an explicit description of the expansion is given. These results were recently applied on infinite-domain constraint satisfaction problems by Bodirsky, Madelaine and Mottet [BMM18].

Curiously, a similar general result in the EPPA context was formulated a lot earlier in the form of the Herwig–Lascar theorem [HL00]. One of the contributions of Theorem 1.5.11 is thus giving a generalisation of the Herwig–Lascar theorem with a new, and more systematic, proof. The Herwig–Lascar theorem has been generally regarded as the deepest result in the area. See also [Bit19].

1.6.5 Classes with algebraic closures

Amalgamation classes which are not strong amalgamation classes lead to Fraïssé limits with non-trivial algebraic closure:

Definition 1.6.3. Let L be a relational language, let \mathbf{A} be an L -structure and let S be a finite subset of A . The *algebraic closure of S in \mathbf{A}* , denoted by $\text{ACl}_{\mathbf{A}}(S)$, is the set of all vertices $v \in A$ for which there is a formula ϕ in the language L with $|S| + 1$ variables such that $\phi(\vec{S}, v)$ is true and there are only finitely many vertices $v' \in A$ such that $\phi(\vec{S}, v')$ is also true. (Here, \vec{S} is a fixed enumeration of S .)

To our best knowledge, the first class with non-trivial algebraic closure studied in the direct structural Ramsey context was the class of all finite bowtie-free graphs [HN18a] (inspired by the aforementioned work of Cherlin, Shelah and Shi [CSS99] and an earlier result of Komj ath, Mekler and Pach [KMP88]).

While this class was originally thought to be a good candidate for a class with no precompact Ramsey expansion, this turned out to not be the case. In the direction of finding a proof of the Ramsey property for this class, the partite construction was extended for languages involving functions, which can be used to connect a set to its closure. This technique turned out to be very general and is discussed in detail in [HN19, Section 4.3].

EPPA for the class of all finite bowtie-free graphs was considered by Siniora [Sin17a, Sin17b] who gave a partial result for extending one partial automorphisms. Thanks to the general results on free amalgamation classes with unary functions the proof of the existence of a Ramsey expansion and of EPPA for the class of all finite bowtie-free graphs is now easy [EHN17, Section 5.4] and in fact generalises to all known Cherlin–Shelah–Shi classes with one constraint [HN19, Section 4.4.2].

Introducing expansions with function symbols has turned out to be a useful tool for many additional examples which we outline next.

1.6.6 Classes with definable equivalences

Example 1.5.15 showing a subclass which is not a locally finite subclass can be generalised using the model-theoretical notion of definable equivalences.

Let \mathbf{A} be an L -structure. An *equivalence formula* is a first order formula $\phi(\vec{x}, \vec{y})$ which is symmetric and transitive on the set of all n -tuples \vec{a} of vertices of \mathbf{A} where $\phi(\vec{a}, \vec{a})$ holds (the set of such n -tuples is called the *domain* of the equivalence formula ϕ).

It is not hard to observe that definable equivalences with finitely many equivalence classes may be obstacles to being Ramsey. For a given ordered structure \mathbf{U} , we say that ϕ is an *equivalence formula on copies of \mathbf{A}* if ϕ is an equivalence formula, $\phi(\vec{a}, \vec{a})$ holds if and only if the structure induced by \mathbf{U} on \vec{a} is isomorphic to \mathbf{A} (in some fixed order of vertices of \mathbf{A}).

Proposition 1.6.4 ([HN19, Proposition 4.25]). *Let \mathcal{K} be a hereditary Ramsey class of ordered L -structures, \mathbf{U} its Fraïssé limit, \mathbf{A} a finite substructure of \mathbf{U} and ϕ an equivalence formula on copies of \mathbf{A} . Then ϕ has either one or infinitely many equivalence classes.*

Proof. Assume to the contrary that ϕ is an equivalence formula on copies of \mathbf{A} which defines k equivalence classes, $1 < k < \infty$. It is well known that from ultrahomogeneity we can assume that ϕ is quantifier-free. Consequently, there is a finite substructure $\mathbf{B} \subseteq \mathbf{U}$ containing two copies of \mathbf{A} which belong to two different equivalence classes of ϕ . Partition $(\frac{\mathbf{U}}{\mathbf{A}})$ to k equivalence classes of ϕ . Since ϕ is quantifier free, we get that there is no $\tilde{\mathbf{B}} \in (\frac{\mathbf{U}}{\mathbf{B}})$ such that $(\frac{\tilde{\mathbf{B}}}{\mathbf{A}})$ would lie in a single equivalence class. Clearly, this implies that there is no $\mathbf{C} \in \mathcal{K}$ such that $\mathbf{C} \rightarrow (\mathbf{B})_k^{\mathbf{A}}$, hence contradicting the Ramsey property. \square

In model theory, an *imaginary* element \vec{a}/ϕ of \mathbf{A} is an equivalence formula ϕ together with a representative \vec{a} of some equivalence class of ϕ . Structure \mathbf{A} *eliminates imaginary \vec{a}/ϕ* if there exists a first order formula $\Phi(\vec{x}, \vec{y})$ such that there is a unique tuple \vec{b} such that $\phi(\vec{x}, \vec{a}) \iff \Phi(\vec{x}, \vec{b})$.

The notion of elimination of imaginaries separates those definable equivalence which are not a problem for Ramsey and EPPA constructions (for example, graphs have a definable equivalence on pairs of vertices with two equivalence classes: edges and non-edges) and those which are problem (such as balls of diameter one in Example 1.5.14).

To give a Ramsey expansion, we thus first want to expand the structure and achieve elimination of (relevant) imaginaries. For a given equivalence formula ϕ with finitely many equivalence classes, it is possible to expand the language by explicitly adding relations representing the individual equivalence classes. For an equivalence formula ϕ with infinitely many equivalence classes, the elimination of imaginaries can be accomplished by adding new vertices representing the equivalence classes and a function symbol linking the elements of individual equivalence classes to their representative.

This technique was first used to obtain Ramsey expansion of S -metric spaces [HN19, Section 4.3.2]. Braunfeld used this technique to construct an interesting Ramsey expansion of Λ -ultrametric spaces [Bra17] (here the Ramsey expansion does not consist of a single linear order of vertices but by multiple partial orders), which was later generalised in Konečný’s master thesis [Kon19a].

Analogous problems need to be solved for the EPPA constructions as well. In the case of equivalences on vertices, this can be accomplished by unary functions in the same way as for Ramsey expansion. A technique to work with equivalence on n -tuples with infinitely many equivalence classes was introduced by Ivanov [Iva15] who observed that a technical lemma on “permomorphisms” [Her98, Lemma 1] can be used to obtain EPPA here. Hubička, Konečný and Nešetřil [HKN19b] generalised this technique to the notion of Γ_L -structures which in turn proved to be useful for more involved constructions.

EPPA for the class of all structures with one quaternary relation defining equivalences on subsets of size two with two equivalence classes is an open problem [EHKN19].

1.6.7 *Classes with non-trivial expansions*

Ramsey expansions are not always obtained by ordering vertices and imaginaries alone. An early example (generic local order) is already discussed in Example 1.4.2. Another example of this phenomenon is the class of two-graphs [Sei73]. *Two-graph \mathbf{H} , corresponding to a graph \mathbf{G}* is a 3-uniform hypergraph created from \mathbf{G} by putting a hyper-edge on every triple containing an odd number of edges.

While there are non-isomorphic graphs such that their two-graphs are isomorphic, it is not hard to show that every Ramsey expansion of two-graphs must fix one particular graph [EHKN19, Section 7]. This means that the class of all finite two-graphs (that is, finite hypergraphs which are two-graphs corresponding to some finite graph \mathbf{G}) is not a locally finite subclass of the class of all finite hypergraphs. Yet, quite surprisingly, this class has EPPA as was recently shown by Evans,

Hubička, Nešetřil and Konečný [EHKN19]. As discussed in Section 1.4 presents interesting example with respect to question Q₄ (where the answer is open but conjectured to be negative) and Q₅ (where answer is negative).

We believe that this class is the first known example of a class with EPPA but not ample generics [EHKN19, Section 8]. It is also presently open if this class has coherent EPPA.

Yet another related example is the class of all semigeneric tournaments. A Ramsey expansion was given by Jasiński, Laflamme, Nguyen Van Thé and Woodrow [JLNVTW14]. EPPA was recently claimed by Hubička, Jahel, Konečný and Sabok [HJKS19a].

These examples share the property that there is no automorphism-preserving completion property, yet EPPA follows by some form of the valuation construction. This suggests that Theorem 1.5.11 can be strengthened. It is however not clear what the proper formulation should be.

Most of these examples can be seen as reducts of other homogeneous structures and the valuation constructions are related to this process. There is an ongoing classification programme of such reducts which makes use of knowing the corresponding Ramsey classes (see e. g. [Tho91, Tho96, JZ08, BCP10, BP11, BPP15, PPP+14]). This may close a full circle: the study of Ramsey properties leads to a better understanding of reducts which, in turn, may lead to a better understanding of EPPA. Some open problems in this direction are discussed in [EHKN19].

Perhaps the most surprising example was however identified by Evans and is detailed in Section 1.7.

1.6.8 *Limitations of general methods*

Current general theorems seems to provide a very robust and systematic framework for obtaining Ramsey expansions and EPPA. We refer the reader to [HN19, Section 4] which provides numerous additional positive examples. However, there are known examples which do not follow by these techniques:

1. The class of all finite groups is known to have EPPA [SS19, Son17]. It is not clear what the corresponding Ramsey results should be. Moreover, EPPA does not seem to follow by application of Theorem 1.5.11.
2. The class of all skew-symmetric bilinear forms has EPPA [Eva05]. Again it is now known if there is a precompact relational Ramsey expansion and Theorem 1.5.11 does not seem to apply.

3. The Graham–Rothschild theorem implies that the class of finite boolean algebras is Ramsey. In general, dual Ramsey theorems seems to not be covered by the presented framework, including structural dual Ramsey theorems by Prömel [Prö85], Frankl, Graham and Rödl [FGR87] and Solecki [Sol10]. These results all share similar nature but differ in the underlying categories. It seems to the author that proper foundations for the structural Ramsey theory of dual structures need to be developed building on non-structural results [Lee73, GLR72, Spe79], approaches based on category theory [Prö85, Section 2.3] and recent progress on the projective Fraïssé limits [IS06, Pan16]. It is also an interesting question, how to relate this to the self-dual Ramsey theorem of Solecki [Sol13, Sol11].

This line of research seems promising. Bartošová and Kwiatkowska applied the KPT-correspondence on the Lelek fans [BK14, BK17]. A dualisation of Theorem 1.5.12 is currently work in progress. Dualisation of EPPA is also a possible future line of research.

4. Melleray and Tsankov adapted the KPT-correspondence to the context of metric structures [MT14]. First successful application of Ramsey theory in this direction was done by Bartošová, Lopez-Abad, Lupini and Mbombo [BLALM17].
5. Sokić has shown Ramsey expansions of the class of lattices [Sok15]. Again it is not clear how to obtain this result by application of Theorem 1.5.12.

1.7 NEGATIVE RESULTS

We conclude this introduction by a brief review of the surprising negative result of [EHN19].

Given the progress of the classification programme, one may ask if there are negative answers to question Q1 (Section 1.4). More specifically, is there a homogeneous structure \mathbf{H} such that there is no Ramsey structure \mathbf{H}^+ which is a precompact relational expansion of \mathbf{H} ?

Eliminating easy counter-examples (such as \mathbb{Z} seen as a metric space [NVT15]), a more specific question was raised by Melleray, Nguyen Van Thé and Tsankov [MNVT15, Question 1.1], asking if the answer to question Q1 is positive for every ω -categorical homogeneous structure \mathbf{H} (that is, it's automorphism group has only finitely many orbits on n -tuples for every n). In a more restricted form, Bodirsky, Pinsker and Tsankov [BPT11] asked if the answer to question Q1 is positive for every structure \mathbf{H} homogeneous in a finite relational language.

A negative answer to the question asked by Melleray, Nguyen Van Thé and Tsankov was given by Evans [Eva] (the latter question remains open). This led to the following:

Theorem 1.7.1 (Evans, Hubička, Nešetřil [EHN19]). *There exists a countable, ω -categorical L -structure \mathbf{H} with the property that if $\Gamma \leq \text{Aut}(\mathbf{H})$ is extremely amenable (in other words, Γ is an automorphism group of a Ramsey structure which is an expansion of \mathbf{H}), then Γ has infinitely many orbits on \mathbf{H}^2 . In particular, there is no precompact expansion of \mathbf{H} whose automorphism group is extremely amenable.*

This is demonstrated on a specific structure \mathbf{H} constructed using the Hrushovski predimension construction. The automorphism group of \mathbf{H} has many interesting properties (see [EHN19]). We single out the fact that, unlike the case of integers (where the only possible Ramsey expansion is the trivial one naming every vertex), there exists a non-trivial Ramsey (non-precompact) expansion \mathbf{H}^+ of \mathbf{H} . In this context, however, the expansion property can not be used to settle that \mathbf{H}^+ is “minimal” in a some sense. Answering a question asked by Tsankov in Banff meeting in 2015, [EHN19] shows that $\text{Aut}(\mathbf{H}^+)$ is maximal among extremely amenable subgroups of $\text{Aut}(\mathbf{H})$. This justifies the minimality of this Ramsey expansion. The question about uniqueness remains open.

A similar situation arises with EPPA. The age of \mathbf{H} does not have EPPA [EHN19, Corollary 3.9] (thus the answer to Q3 is negative). However, a meaningful EPPA expansion is proposed in the form of an amenable subgroup [EHN19, Theorem 6.11]. This subgroup is again conjectured to be maximal among the amenable subgroups of $\text{Aut}(\mathbf{H})$ [EHN19, Conjecture 7.5]. Hubička, Konečný and Nešetřil recently proved EPPA

for this class [HKN19a, Theorem 9.8]. The second part of [EHN19, Conjecture 7.5] (about maximality) remains open.

These examples are not isolated and several variants of this construction are considered [EHN19]. This shows that the interplay between EPPA, Ramsey classes and amalgamation classes is more subtle than previously believed, especially in the context of structures in languages with functions. Because most of structural Ramsey theory was developed in the context of relational structures, these situations have only been encountered recently. These results also demonstrate that it is meaningful (and very interesting) to extend the classification programme even for structures with such behaviour.

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APPENDIX A: Jan Hubička and Jaroslav Nešetřil. Universal structures with forbidden homomorphisms. In Å. Hirvonen, J. Kontinen, R. Kossak, and A. Villaveces, editors, *Logic Without Borders: Essays on Set Theory, Model Theory, Philosophical Logic and Philosophy of Mathematics*, Ontos Mathematical Logic Series, pages 241–264. De Gruyter, 2015.

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