Milliken tree theorem and big Ramsey degrees of graphs and hypergraphs

Jan Hubička

Department of Applied Mathematics
Charles University
Prague

Infinite Ramsey Theory, 2025, Banff

Miss you, Robert!

Proof techniques

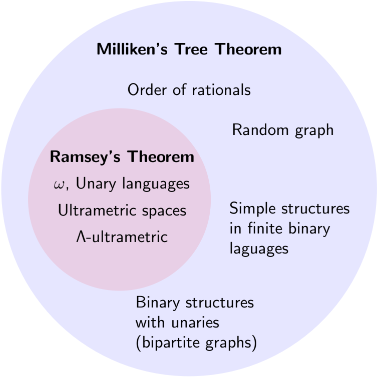
Ramsey's Theorem

ω, Unary languages

Ultrametric spaces

Λ-ultrametric

Proof techniques



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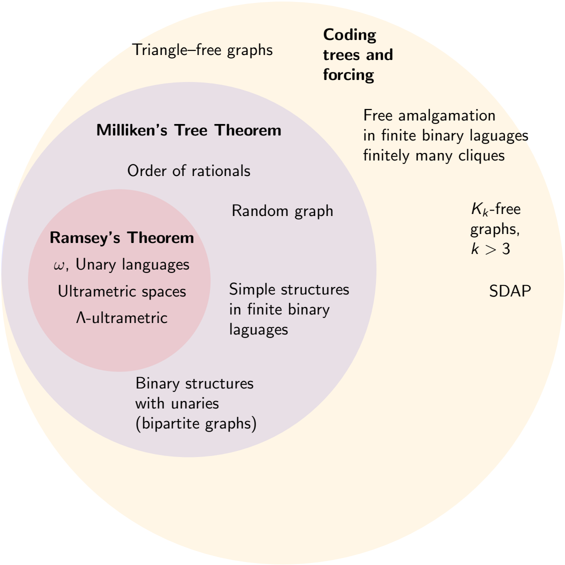

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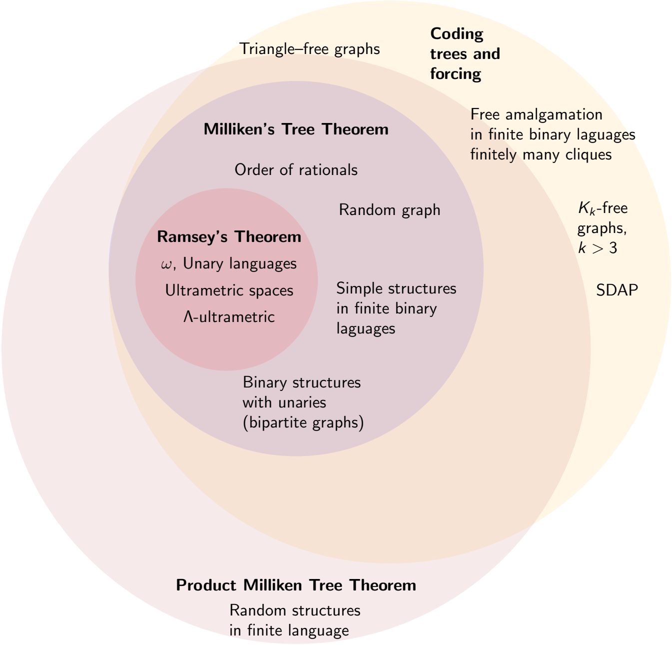

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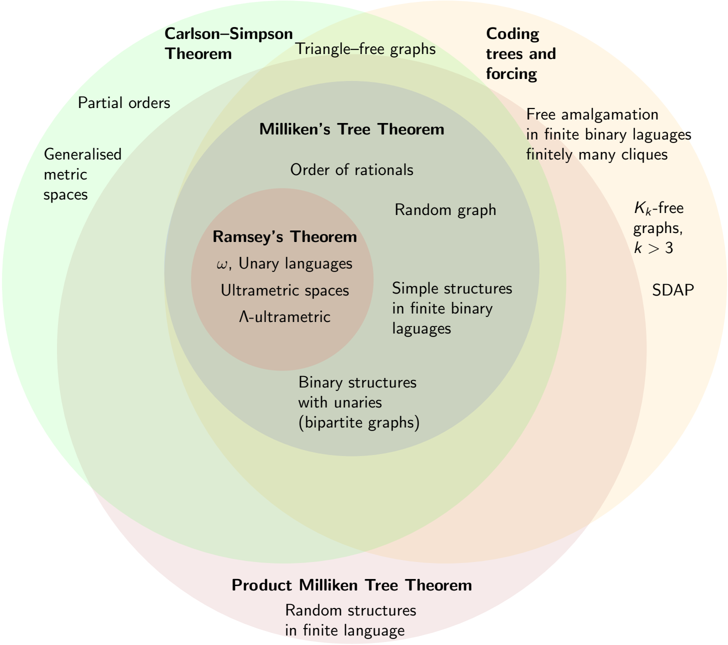

Ordered hypergraphs and type-equivalence

For simplicity, I will speak only of u-uniform hypergraphs with $u = 2$ (graphs) and $u = 3$. All finite or countable. Everything extends naturally to relational structures.

1. Hypergraph $\mathbf{H}$ has vertex set H and edges $E_{\mathbf{H}}$.
2. Hypergraph $\mathbf{H}$ is enumerated if $H = |H| = \{0, 1, \ldots, |H| - 1\}$.
3. Embeding $f\colon \mathbf{H} \to \mathbf{H}'$ is order-preserving if $\forall_{u,v \in H} : u < v \implies f(u) < f(v)$.
4. $\mathbf{H}$ is order-isomophic to $\mathbf{H}'$, $H \simeq H'$, if there is order-preserving isomorphism $\mathbf{H} \to \mathbf{H}'$
5. Given $S \subseteq H$, $\mathbf{H} \restriction_S$, is hypergraph induced by $\mathbf{H}$ on S.
6. I identify $n \in \omega$ with $\{0, 1, \ldots, n - 1\}$.

Definition (Type-equivalence)

Let $\mathbf{H}$ be enumerated hypergraph, $\ell \in H$, and $S, S' \subseteq H \setminus \ell$.
S and S' have the same type over ℓ, and write $S \simeq^{\ell}_{\mathbf{H}} S'$, if and only if $\mathbf{H} \restriction_{\ell \cup S} \simeq \mathbf{H} \restriction_{\ell \cup S'}$.

Type-respecting embeddings

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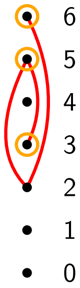
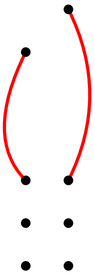


$\{5\} \simeq^{3}_{\mathbf{H}} \{6\}$



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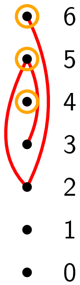

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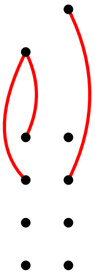

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Definition (Meets (splits))

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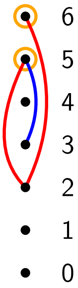

$\{5\} \wedge_{\mathbf{H}} \{6\} = 3$

$\{1\} \wedge_{\mathbf{H}} \{2\} = 1$

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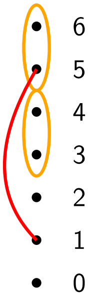

for graphs

$\{u_0 < u_1\} \wedge_{\mathbf{H}} \{v_0 < v_1\}$

is

$\min(\{u_0\} \wedge_{\mathbf{H}} \{v_0\}, \{u_1\} \wedge_{\mathbf{H}} \{v_1\})$

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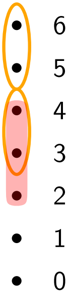

$\{3,4\} \wedge_{\mathbf{H}} \{5,6\} = 2$

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$S \wedge_{\mathbf{H}} S'$ is defined only if $\mathbf{H} \restriction S$ is order-isomorphic to $\mathbf{H} \restriction_{S'}$ (i.e. $S \simeq^{\mathbf{H}}_{0} S'$).

Definition (Type-resepecting embedding)

Let $\mathbf{H}$, $\mathbf{H}'$ be odered hypergraphs.
Embedding $f : \mathbf{H} \to \mathbf{H}'$ is type-resepecting if it is order-preserving and

$$\forall_{S, S' \subseteq H} : S \simeq^{\mathbf{H}}_{0} S' \implies f(S \wedge_{\mathbf{H}} S') = f[S] \wedge_{\mathbf{H}'} f[S'].$$

All known big Ramsey results are Ramsey theorems on type-respecting embeddings.

Ramsey theorem for type-respecting embeddings

Given enumerated hypergraph $\mathbf{H}$ and $\mathbf{H}'$, $n \in \mathbf{H}$ we denote by $\mathrm{TREmb}_n(\mathbf{H}, \mathbf{H}')$ the set of all type-respecting embeddings $\mathbf{H} \to \mathbf{H}'$ which are identity when restricted to n.

We prove:

Theorem

Let $\mathbf{R}$ be the Rado graph (i.e. universal and homogeneous 2-uniform hypergraph). Then for every finite enumerated hypergraph $\mathbf{A}$, every $n < |A|$ and every coloring

$$\chi : \mathrm{TREmb}_n(\mathbf{A}, \mathbf{R}) \to 2$$

there exists $f \in \mathrm{TREmb}_n(\mathbf{R}, \mathbf{R})$ so that χ is constant on

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Generalises to u-uniform hypergraphs and relational structures.

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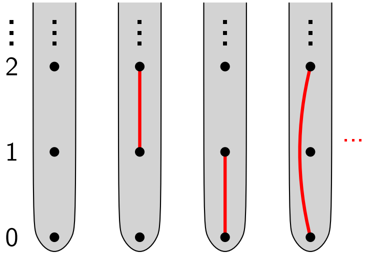
M Balko, D Chodounský, J Hubička, M Konečný, L Vena: Big Ramsey degrees of 3-uniform hypergraphs are finite, Combinatorica 42 (2022), 659–672.

Amalgamating all enumerations

Take all countable, enumerated graphs and amalgamate them over common initial parts.



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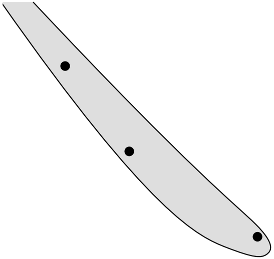

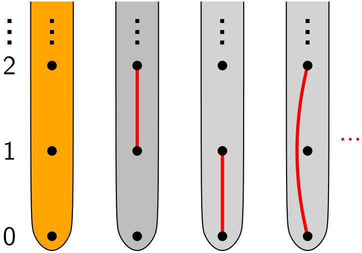

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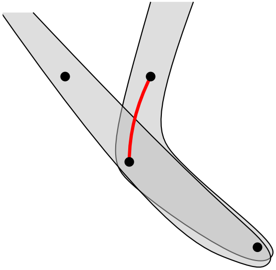

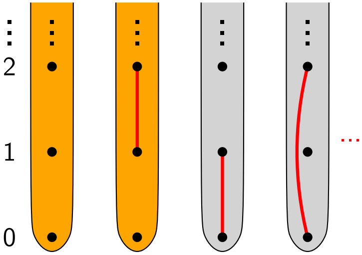

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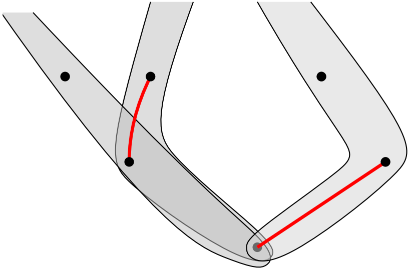

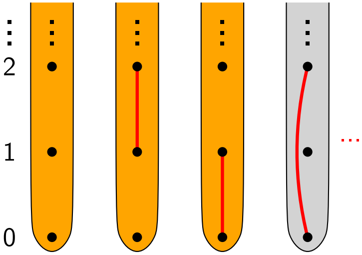

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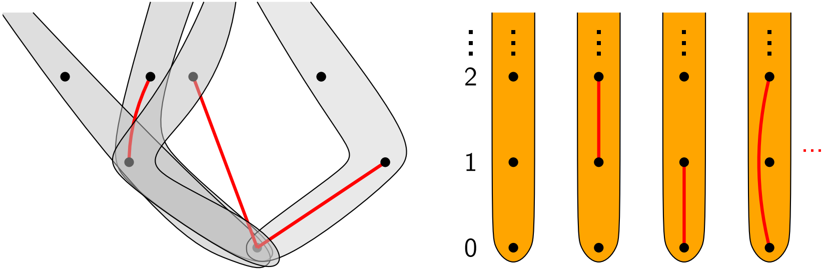

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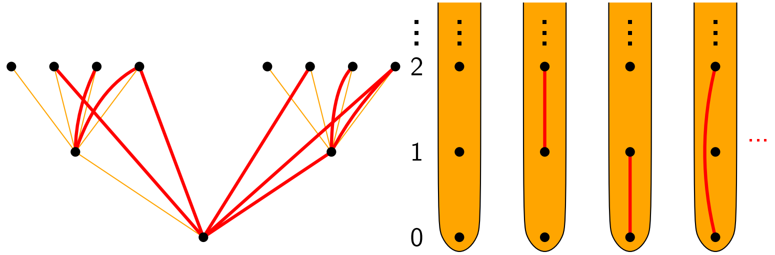

Given graphs **A**, **B** we write $\mathbf{A} \subseteq \mathbf{B}$ if **A** is induced subgraph of **B**.

Definition (Graph $\mathbf{K}_I$)

Given finite enumerated graph **I** and denote by $\mathbf{K_I}$ graph with

1. Vertices of $\mathbf{K_I}$ are all non-empty finite enumerated graphs **A** satisfying $\mathbf{A} \restriction_{|I|} = \mathbf{I}$, and graphs $\mathbf{I} \restriction_n$, $1 \leq n < |I|$.
2. $\mathbf{A} \subseteq \mathbf{B}$ forms an edge iff $\{\max A, \max B\} \in E_{\mathbf{B}}$.
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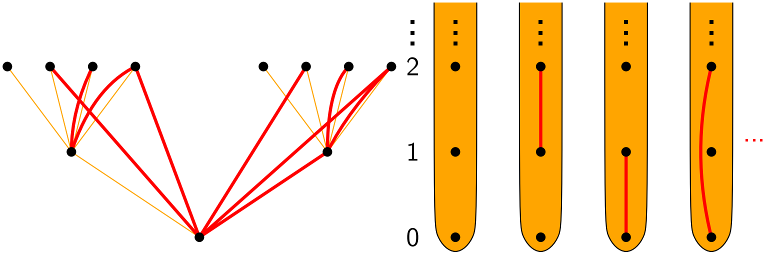

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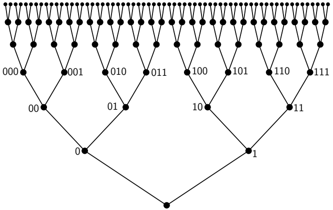
Goal: Ramsey theorem for $\mathbf{K_I}$

Trees

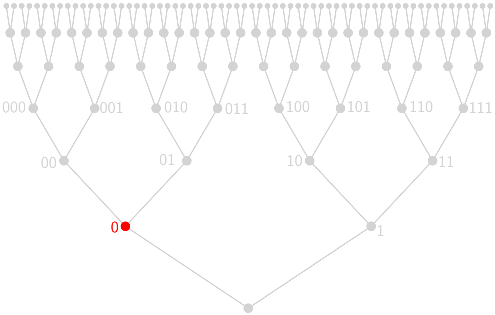
Trees (terminology)



$(2^{<7}, \sqsubseteq)$

- A tree is a (possibly empty) partially ordered set $(T, <_T)$ such that, for every node $t \in T$ its downset $\{ s \in T : s <_T t \}$ is finite and linearly ordered by $<_T$.
- Tree is rooted if it has a unique minimal element (root).
- Level of a node $t \in T$ is $|t|_T = |\{ s \in T : s <_T t \}|$.
- $T(n) = \{ t \in T : |t|_T = n \}$.
- $T(<n) = \{ t \in T : |t|_T < n \}$.
- For $s, t \in T$, the meet $s \wedge_T t$ of s and t is the largest $s' \in T$ such that $s' \leq_T s$ and $s' \leq_T t$.
- The height of T is the $h \in \omega + 1$ such that $T(h) = \emptyset$.
- A subtree of a tree T is a subset $S \subseteq T$ with the induced partial ordering.
- The node s is an immediate successor of t in T if $t <_T s$ and there is no $s' \in T$ such that $t <_T s' <_T s$.
- Node with no successors is leaf.

Strong subtree



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1. S is closed for meets. (In particular, S is rooted.)

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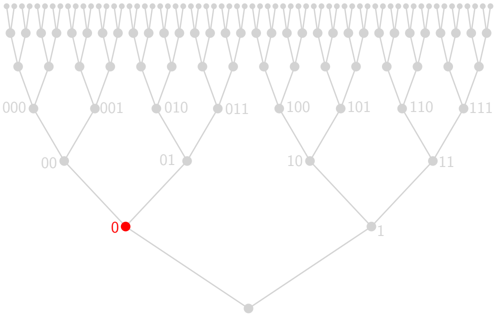

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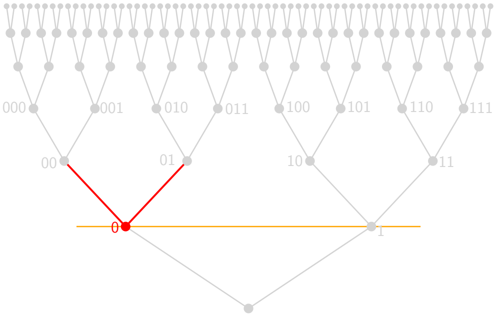

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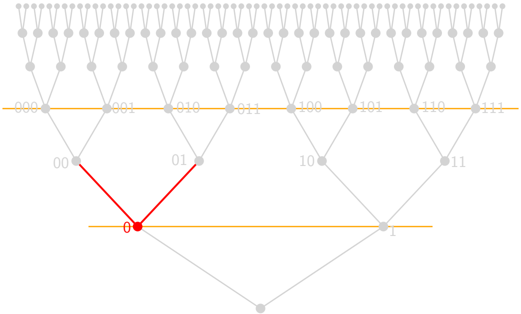

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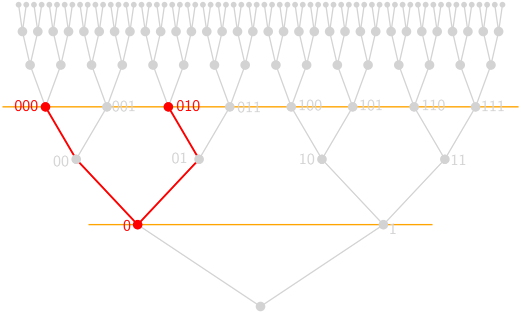

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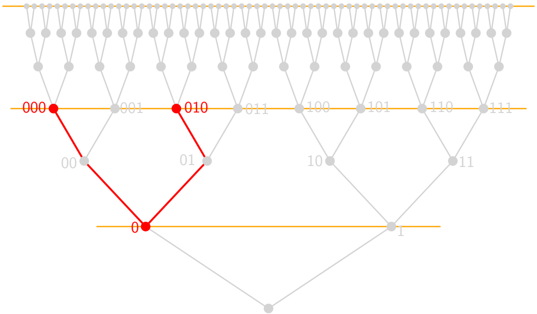

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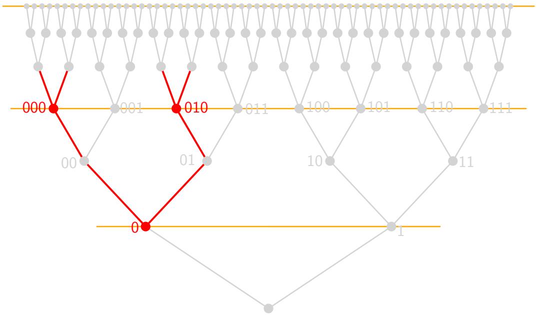

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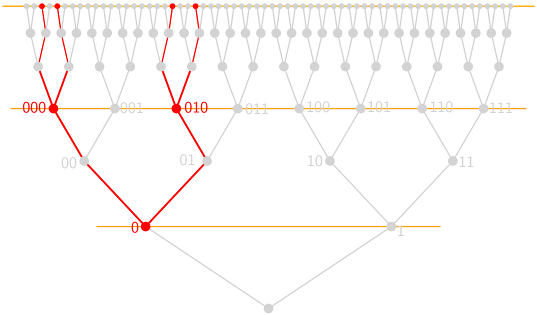

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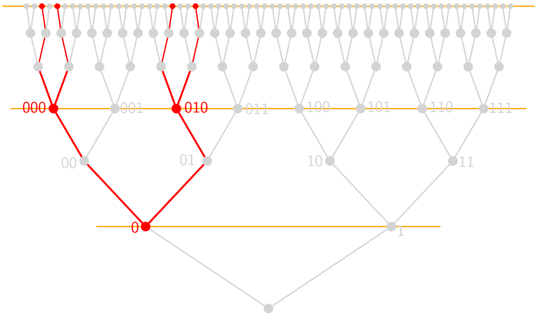

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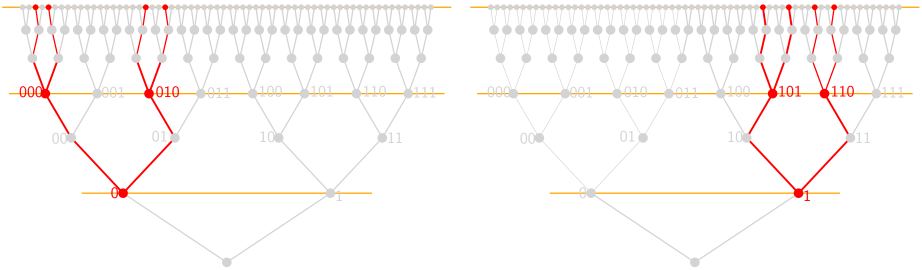
2. For every $a \in S(<(n-1))$ and every immediate successor b of a in T there is an unique immediate successor c of a in S such that $a \sqsubseteq b \sqsubseteq c$. (If $n = \omega$ then every $a \in S$.)
3. S is level preserving: Every level of S is a subset of some level of T.
4. S has height n.

Product tree

A product (vector) tree $\mathbf{T} = (T_i)_{i<n}$ is a finite sequence of n trees.

A (product) strong subtree $\mathbf{S} = (S_i)_{i<n}$ is a sequence such that for every $i < n$ tree S_i is a strong subtree of T_i and all the subtrees share the same levels.

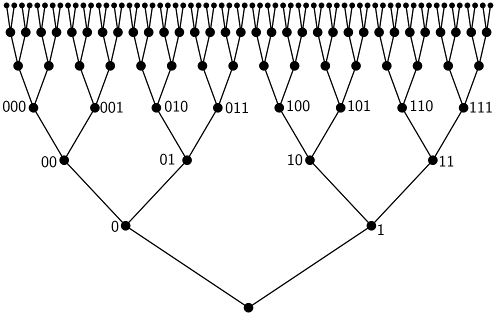


Ramsey-type theorem for strong subtrees

Let $\mathbf{T}$ be a (product) tree and $\ell \leq k \in \omega + 1$. We use $\mathrm{Str}_{k,\ell}(\mathbf{T})$ to denote the set of all strong subtrees of T of height k which include first ℓ levels.

Theorem (Milliken 1979)

For every (product) $\mathbf{T}$ *tree consisting of rooted finitely branching trees with no leaves, every* $\ell < k \in \omega$ *and every finite colouring of* $\mathrm{Str}_{k,\ell}(\mathbf{T})$ *there is* $\mathbf{S} \in \mathrm{Str}_{\omega,\ell}(\mathbf{T})$ *such that the set* $\mathrm{Str}_{k,\ell}(\mathbf{S})$ *is monochromatic.*



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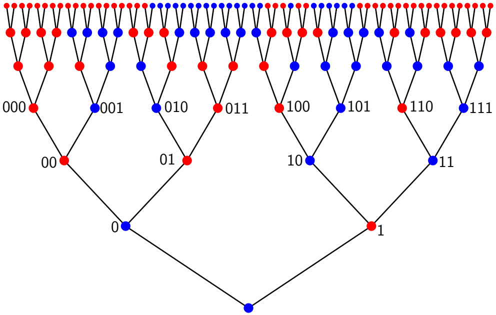

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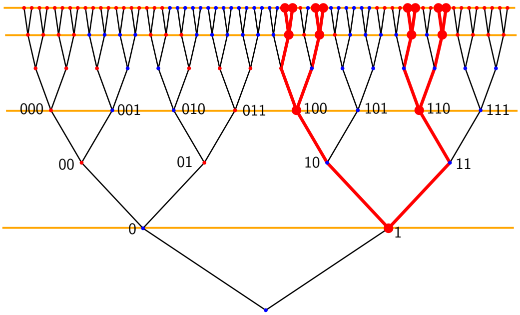

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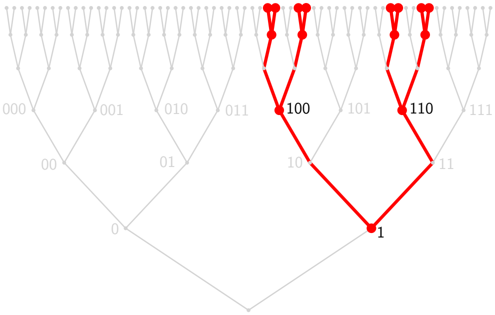

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Idea: Get a correspondence of strong subtrees and type-respecting embeddings to $\mathbf{K_I}$. Ideally 1–to–1.

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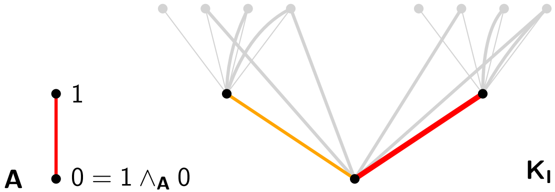

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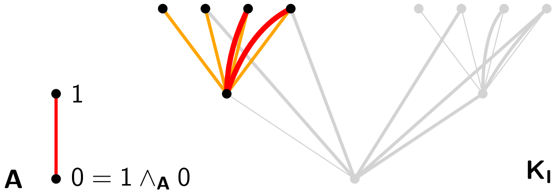

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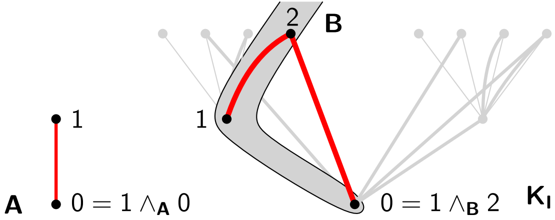

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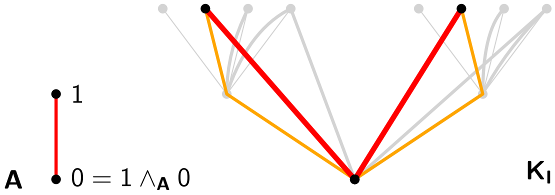

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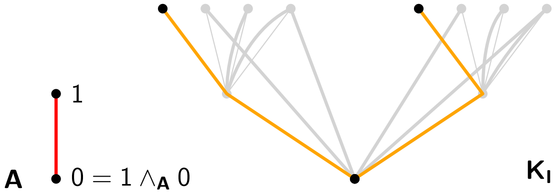

If in trouble, plant more trees

Trees of enumerations and trees of types

Definition (1-type hypergraph)

Given $\ell\omega$, we call a hypergraph $\mathbf{B}$ a 1-type hypergraph of level ℓ if its vertex set is $\ell \cup \{t\}$ (t is a special type vertex) and every edge of $\mathbf{B}$ contains t.

1. Let T_0 be the set of all enumerated graphs. $(T_0, \subseteq)$ is tree of enumerations.
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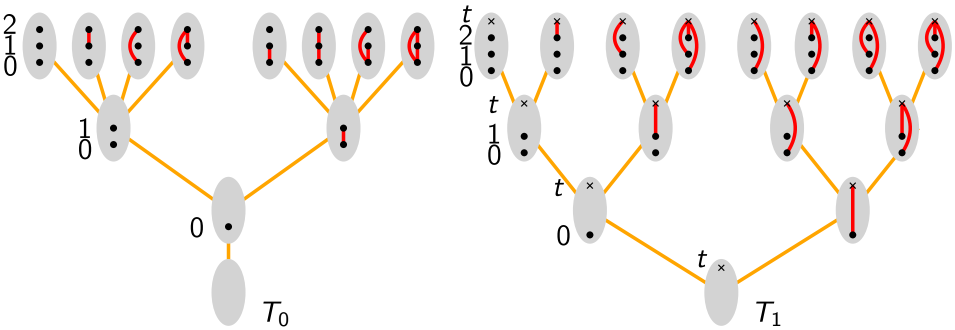

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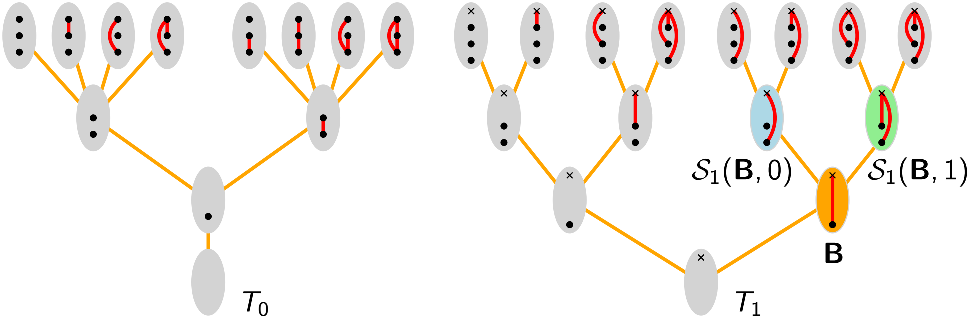

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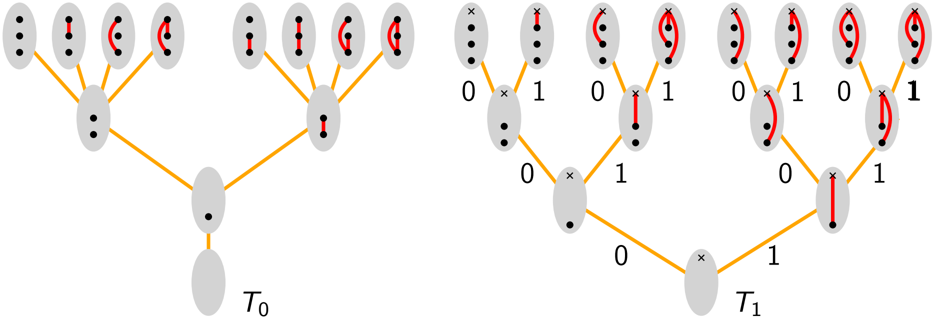

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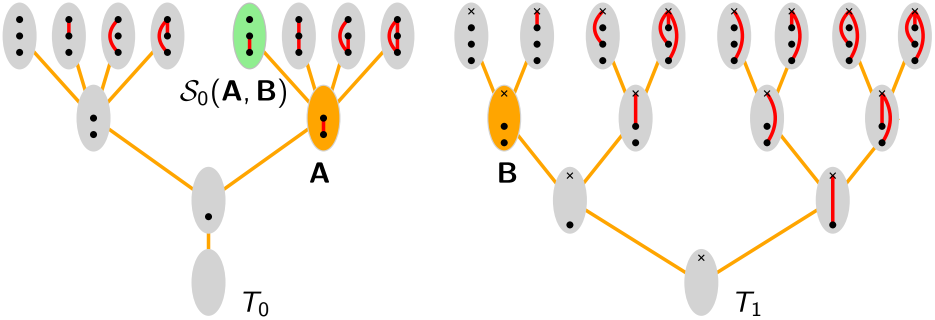

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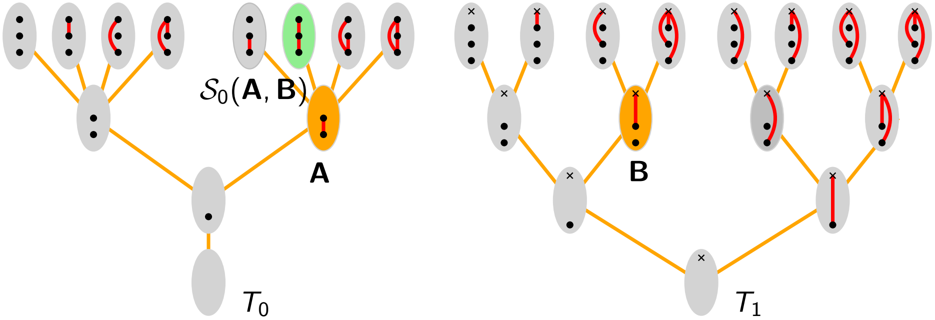

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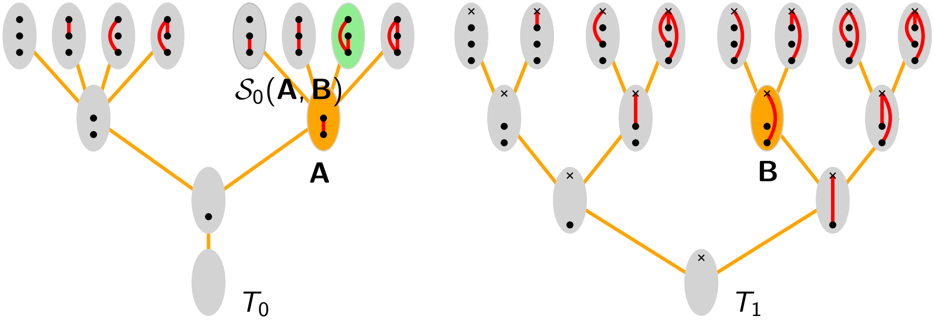

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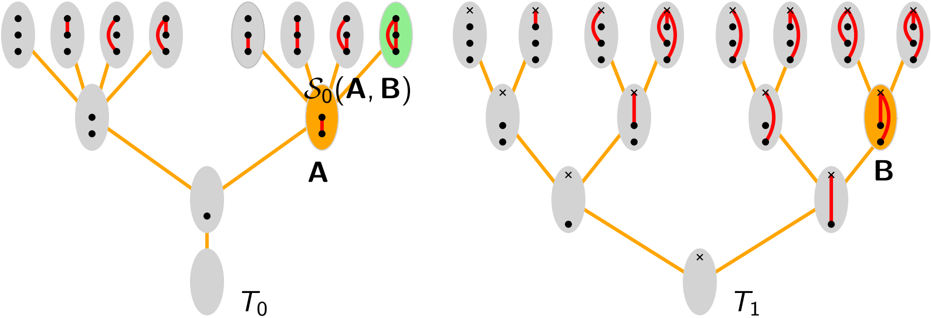

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Definition (Successor operations)

1. Given $\ell \in \omega$, $\mathbf{B} \in T_1(\ell)$ and $e \in \{0, 1\}$ we denote by $\mathcal{S}_1(\mathbf{B}, 1)$ the graph $\mathbf{B}' \in T_1$ with

$$B' = B \cup \{\ell\} \text{ and } E_{\mathbf{B}'} = \begin{cases} E_{\mathbf{B}} & \text{if } e = 0, \\ E_{\mathbf{B}} \cup \{\{\ell, t\}\} & \text{if } e = 1. \end{cases}$$

2. Given $\ell \in \omega$, $\mathbf{A} \in T_0(\ell)$ and $\mathbf{B} \in T_1(\ell)$ we denote by $\mathcal{S}_0(\mathbf{A}, \mathbf{B})$ a graph $\mathbf{A}' \in T_0$ with

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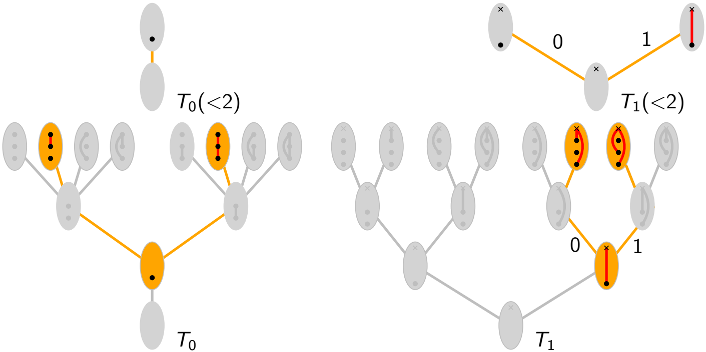
Strong subtrees and canonical embeddings



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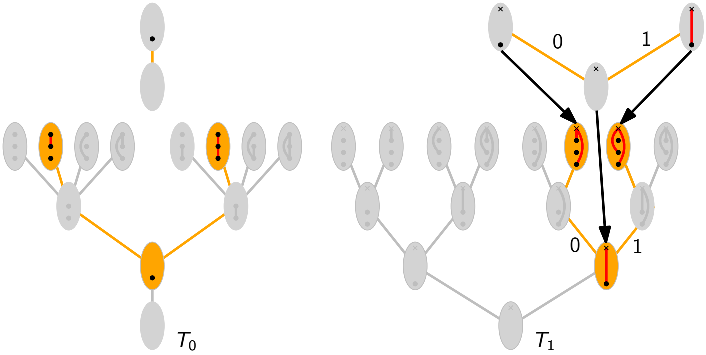

Definition (Canonical embedding $f_1^{\mathbf{T}'}$)

Given $h \in \omega + 1$, $\mathbf{T}' = (T'_0, T'_1) = \mathrm{Str}_{h,0}(\mathbf{T})$, denote by $f_1^{\mathbf{T}'}$ the unique embedding $T_1(<h) \to T'_1$ that is

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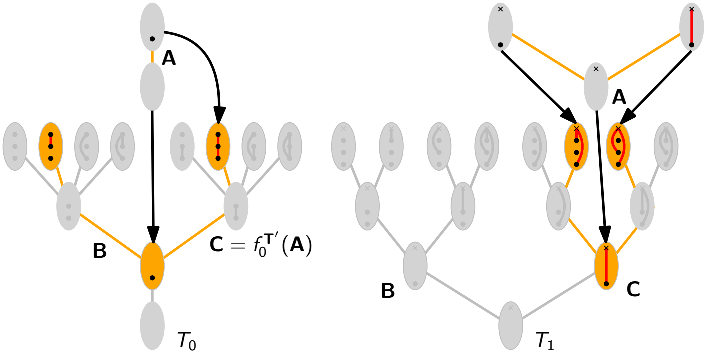

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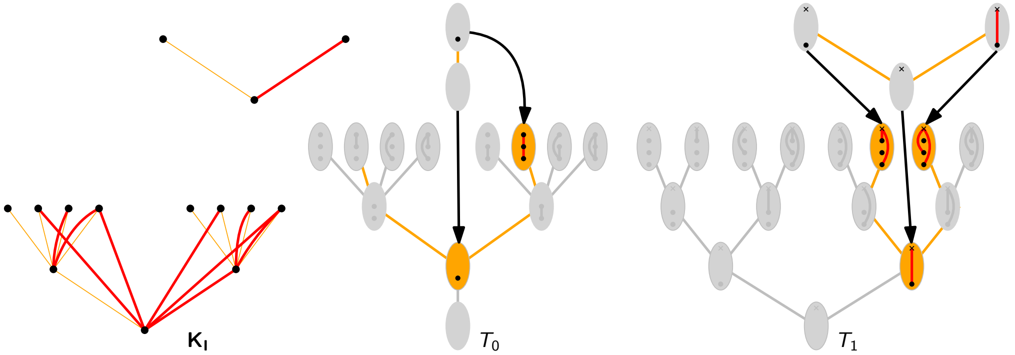
Strong subtrees and canonical embeddings



Recall: Vertices of $\mathbf{K_I}$ are all non-empty finite enumerated graphs $\mathbf{A}$ satisfying $\mathbf{A}\restriction_{|I|} = \mathbf{I}$, and graphs $\mathbf{I}\restriction_n$, $1 \leq n < |I|$.

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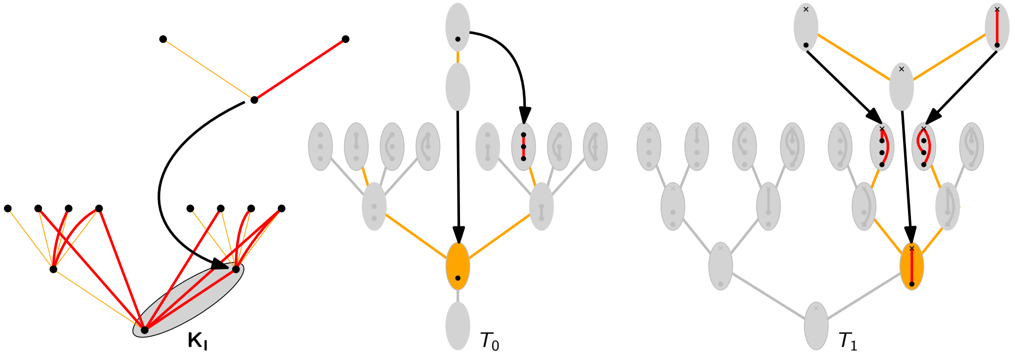
for every $\mathbf{A} \in T_0(<h), \mathbf{B} \in T_1(<h)$ such that $\mathcal{S}_0(\mathbf{A}, \mathbf{B}) \in K_{\mathbf{I}}$.

Strong subtrees and canonical embeddings



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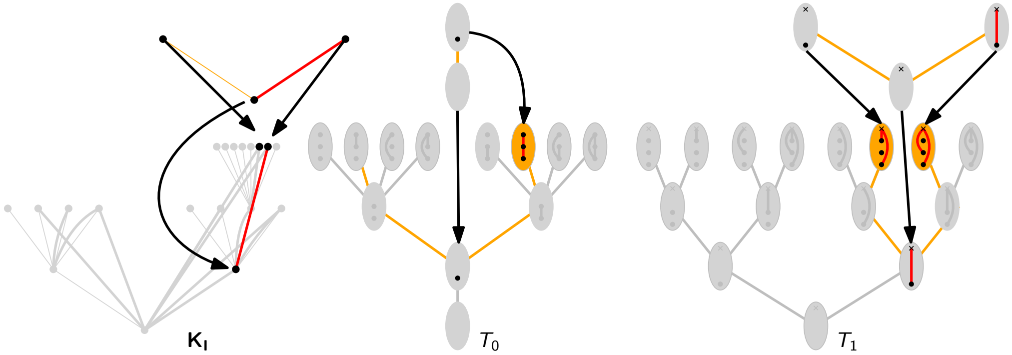

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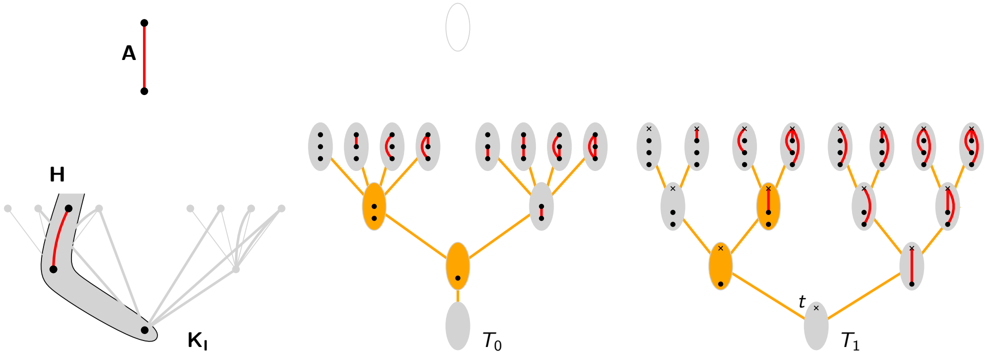

Given enumerated hypergraph $\mathbf{A}$ extending hypergraph $\mathbf{I}$ we denote by $g_{\mathbf{A}}$ the embedding $\mathbf{A} \to \mathbf{K_I}$ defined by $g_{\mathbf{A}}(v) = \mathbf{A} \restriction_{v+1}$.

Lemma

Let $\mathbf{I} \subseteq \mathbf{A}, \mathbf{H}$ *be enumerated graphs. Then for every type-respecting embedding* $e : \mathbf{A} \to \mathbf{H}$ *exists* $\mathbf{T}' \in \mathrm{Str}_{|A|,|I|}(\mathbf{T})$ *such that* $g_{\mathbf{H}} \circ e = f^{\mathbf{T}'} \circ g_{\mathbf{A}}$.

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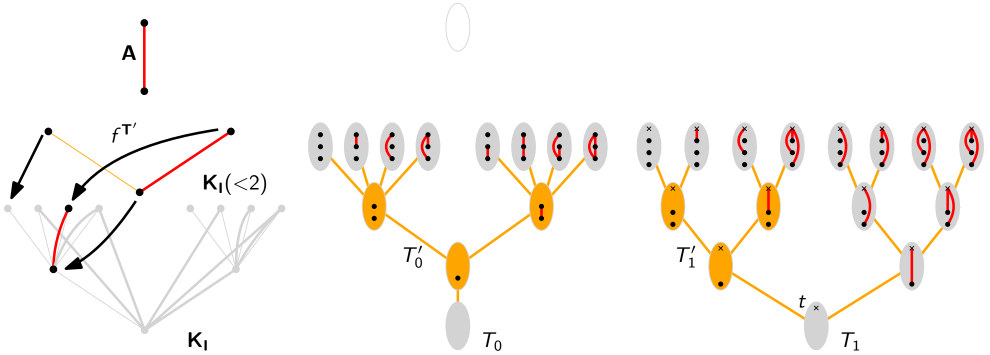

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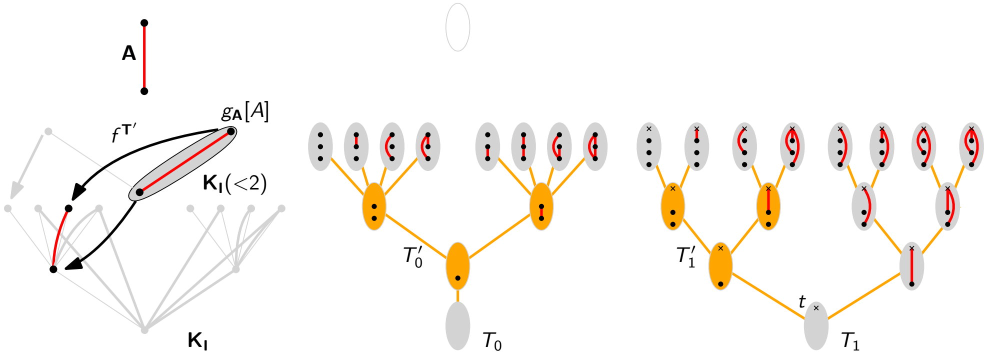
Let $\mathbf{I} \subseteq \mathbf{A}, \mathbf{H}$ *be enumerated graphs. Then for every type-respecting embedding* $e : \mathbf{A} \to \mathbf{H}$ *exists* $\mathbf{T}' \in \mathrm{Str}_{|A|,|I|}(\mathbf{T})$ *such that* $g_{\mathbf{H}} \circ e = f^{\mathbf{T}'} \circ g_{\mathbf{A}}$.

Strong subtrees and canonical embeddings



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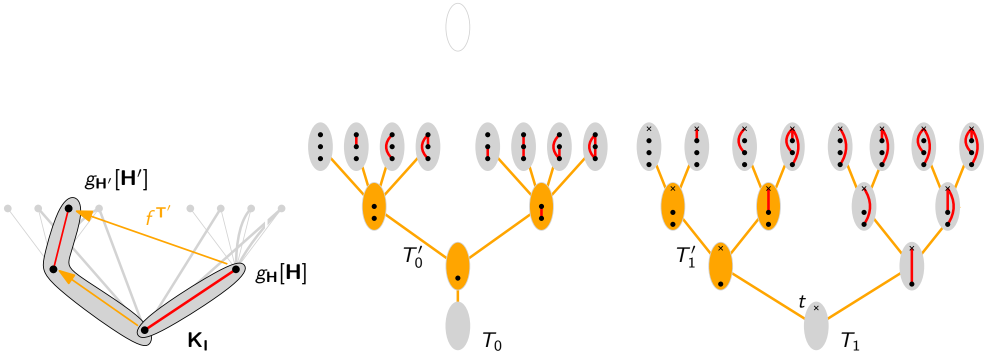

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Lemma

For every $\mathbf{T}' \in \mathrm{Str}_{\infty,|I|}$ and every enumerated hypergraph $\mathbf{H}$ extending $\mathbf{I}$ there exists enumerated hypergraph $\mathbf{H}'$ extending $\mathbf{I}$ such that $g_{\mathbf{H}'}^{-1} \circ f^{\mathbf{T}'} \circ g_{\mathbf{H}}$ is a type respecting embedding $\mathbf{H} \to \mathbf{H}'$.

Putting it all together

Finally, we can show:

Theorem

Let $\mathbf{R}$ *be the Rado graph (i.e. universal and homogeneous 2-uniform hypergraph). Then for every finite enumerated hypergraph* $\mathbf{A}$*, every* $n < |A|$ *and every coloring*

$$\chi : \mathrm{TREmb}_n(\mathbf{A}, \mathbf{R}) \to 2$$

there exists $f \in \mathrm{TREmb}_n(\mathbf{R}, \mathbf{R})$ *so that* χ *is constant on* $f \circ \mathrm{TREmb}_n(\mathbf{A}, \mathbf{R})$

Proof.

1. Fix $\mathbf{A}$, $n < |A|$ and $\chi : \mathrm{TREmb}_n(\mathbf{A}, \mathbf{R}) \to 2$.
2. Put $\mathbf{I} = \mathbf{R} \restriction_n$.
3. Let $\varphi : \mathbf{K_I} \to \mathbf{R}$ be a an embedding extending $g_{\mathbf{I}}^{-1}$ such that for every enumerated hypergraph $\mathbf{H} \supseteq \mathbf{I}$ function $\varphi \circ g_{\mathbf{H}}$ is type-respecting embedding $\mathbf{H} \to \mathbf{R}$.
(exists by the extension property)
4. Define coloring $\chi' : \mathrm{Str}_{|A|,|I|}(\mathbf{T}) \to 2$ by putting $\chi'(\mathbf{T}') = \chi(\varphi \circ f^{\mathbf{T}'} \circ g_{\mathbf{A}})$.
5. By Milliken tree theorem obtain $\mathbf{T}' \in \mathrm{Str}_{\infty,|I|}$.
6. Put $f = \varphi \circ f^{\mathbf{T}'} \circ g_{\mathbf{R}}$.

□


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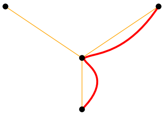
3-uniform hypergraphs



Initial part of 3-uniform hypergraph $\mathbf{K_I}$.

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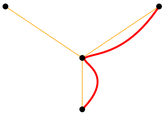

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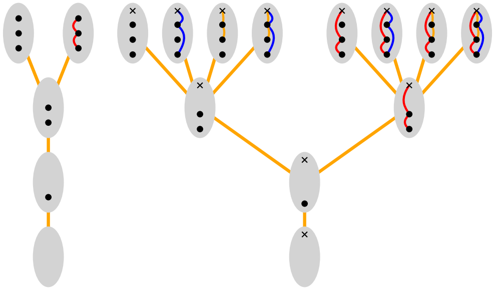
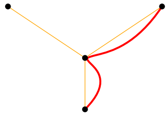


Trees T_0 (enumerations) and T_1 (1-types)

3-uniform hypergraphs



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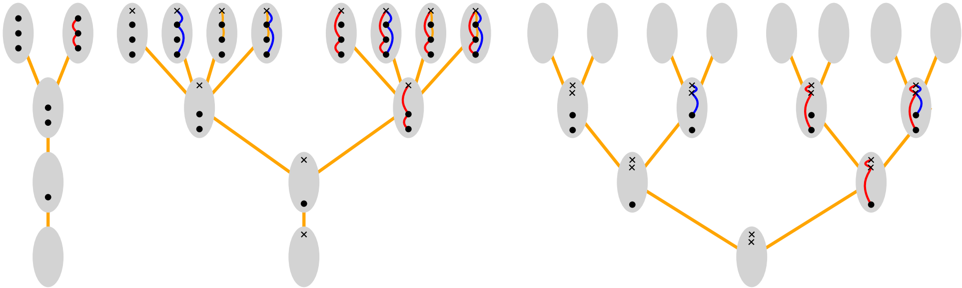

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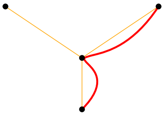

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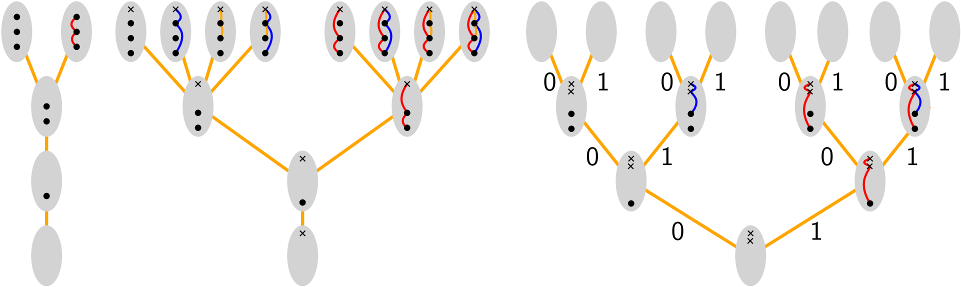

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Successor operations and the rest of the proof can be structured analogously to the binary case.

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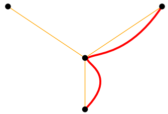

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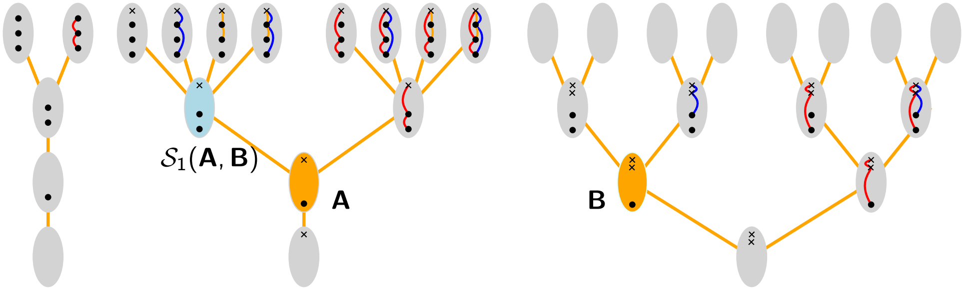

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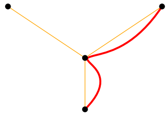

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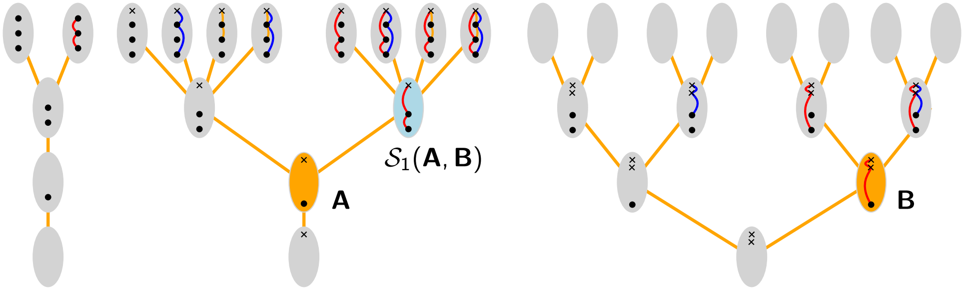

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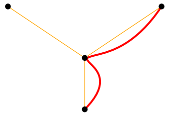

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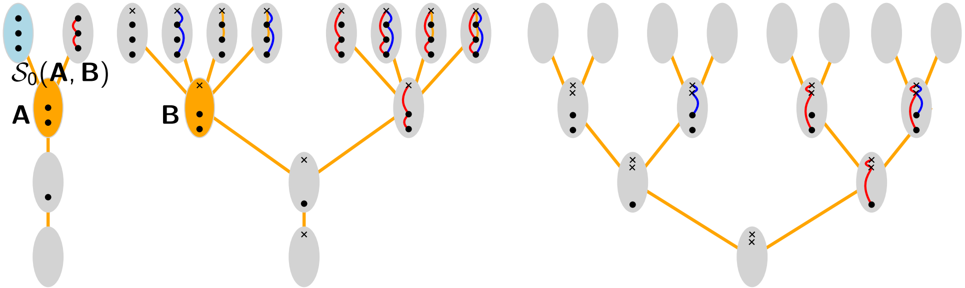

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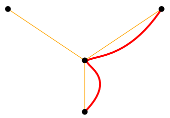

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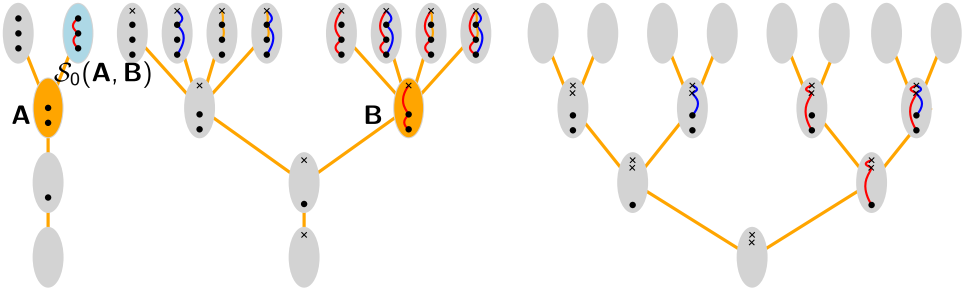

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  - S. Braunfeld, D. Chodounský, N. de Rancourt, J. Hubička, J. Kawach, M. Konečný: Big Ramsey degrees and infinite languages, Advances in Combinatorics, 2024.

Ramsey theorem for 3-uniform hypergraphs

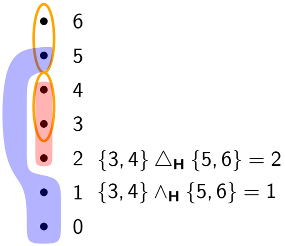
Definition (Aux-type equivalence and meets)

Given an enumerated hypergraph $\mathbf{H}$, vertex $\ell \in H$, and sets $S, S' \in \binom{H\setminus\ell}{2}$ we say that S and S' have the same aux-type over ℓ, and write $S \equiv^{\ell}_{\mathbf{H}} S'$, if and only if for every $v < \ell$ we have

$$\{v\} \cup S \in E_{\mathbf{H}} \iff \{v\} \cup S' \in E_{\mathbf{H}}.$$

We also put

$$S \triangle_{\mathbf{H}} S' = \max\{\ell \in H : \ell \leq \min(S \cup S') \text{ and } S \equiv^{\ell}_{\mathbf{H}} S'\}.$$



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Definition (Aux-type-resepecting embedding)

Given enumerated hypergraphs $\mathbf{H}$ and $\mathbf{H}'$ we call embedding $f : \mathbf{H} \to \mathbf{H}'$ aux-type-resepecting if

1. f is order-preserving,
2. for every $u, v \in H$ it holds that $f(u \wedge_{\mathbf{H}} v') = f(u) \wedge_{\mathbf{H}'} f(v')$, and
3. for every $S, S' \in \binom{H}{2}$ it holds that $f(S \triangle_{\mathbf{H}} S') = f[S] \triangle_{\mathbf{H}'} f[S']$.

Ramsey theorem for 3-uniform hypergraphs II

Given enumerated hypergraphs $\mathbf{H}$ and $\mathbf{H}'$, $n \in \mathbf{H}$ we denote by $\mathrm{ATREmb}_n(\mathbf{H}, \mathbf{H}')$ the set of all aux-type-respecting embeddings $\mathbf{H} \to \mathbf{H}'$ which are identity when restricted to n.
We proved:

Theorem (H., Konečný, Zucker, 2025+)

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Observation

Every aux-type-respecting embedding is also type-respecting.

For every $S, S' \in \binom{H}{2}$ it holds that

$$S \wedge_{\mathbf{D}} S' = \min(\{S \triangle_{\mathbf{D}} s\}, \min S \wedge_{\mathbf{D}} \min S', \max S \wedge_{\mathbf{D}} \max S').$$

A corollary (for graphs)

Lemma (Type stabilization lemma)

Let $\mathbf{R}$ *be an enumerated Rado graph and* $\mathbf{K}$ *an enumerated graph.*
For every embedding $f : \mathbf{R} \to \mathbf{K}$ *there exists type-respecting embedding* $g : \mathbf{R} \to \mathbf{R}$ *such that* $f \circ g$ *is order-preserving and for every* $0 < \ell \leq u < v$

$$u \simeq^{\ell}_{\mathbf{R}} v \implies f(g(u)) \simeq^{f(g(\ell-1))+1}_{\mathbf{K}} f(g(v)).$$

"Every embedding of the Rado graph into an enumerated graph can be turned into an almost type-respecting embedding"

Definition (Recall: Type-resepecting embedding)

Let $\mathbf{H}$, $\mathbf{H}'$ be odered graphs.
Embedding $f : \mathbf{H} \to \mathbf{H}'$ is type-resepecting if it is order-preserving and

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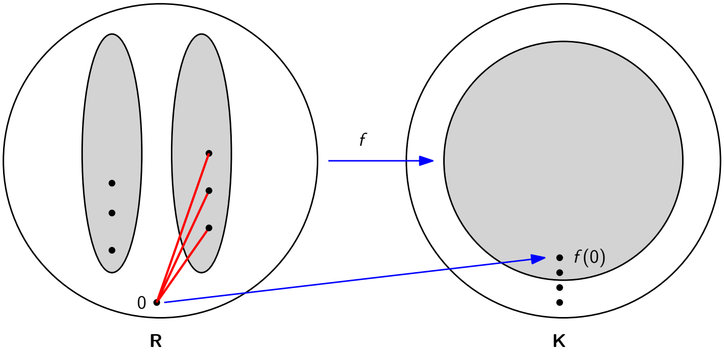

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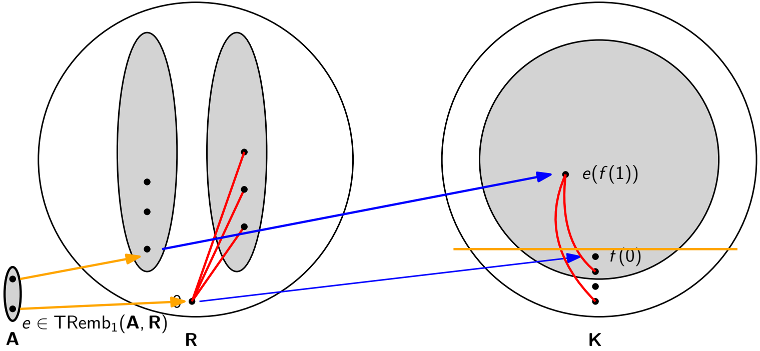

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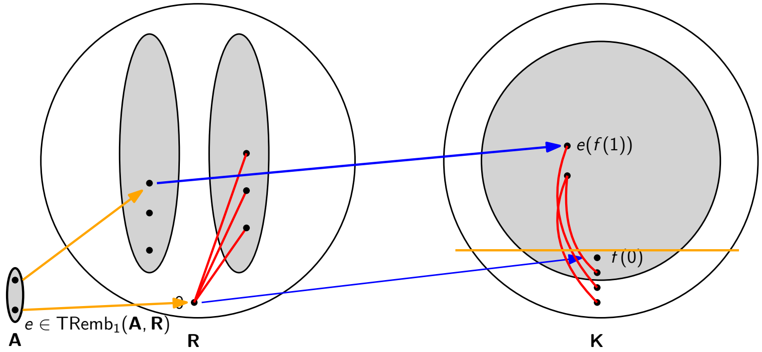

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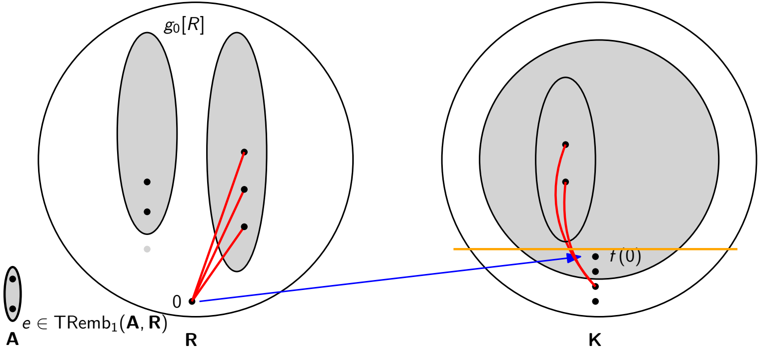

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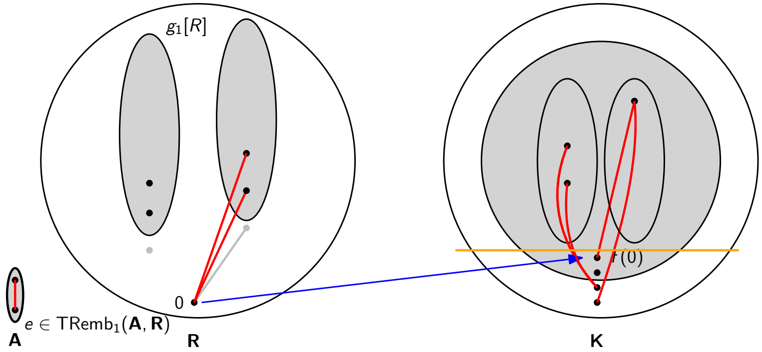

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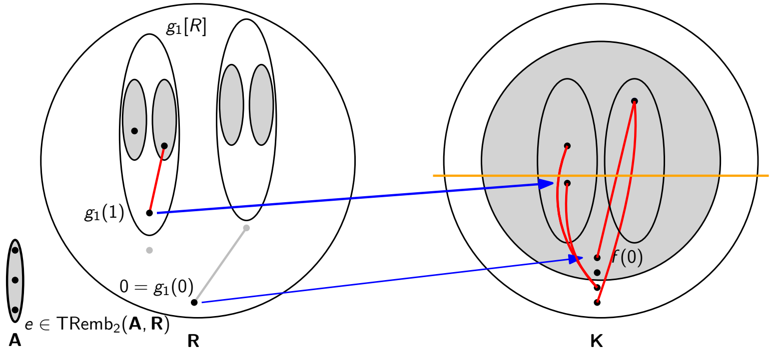

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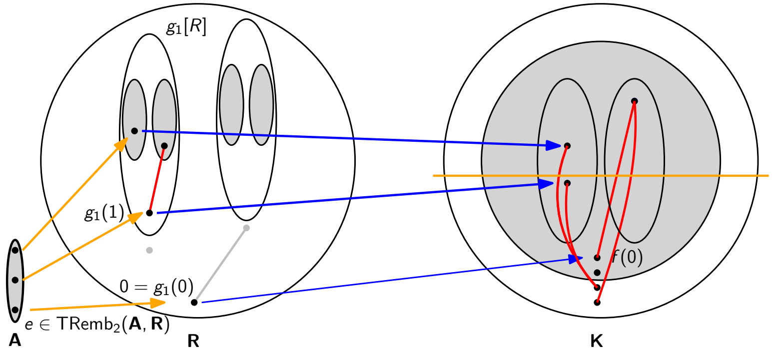

Diaries

Graph daries

Definition (Graph diary)

A graph diary is an enumerated hypergraph **D** such that:

- Every vertex $\ell \in D$ has precisely one of the following roles:
  - **1-type split:**
    - There exist two leaf vertices $u, v > \ell$ such that $\ell = u \wedge_{\mathbf{D}} v$.
    - Whenever $\{\ell, v\}$ is and edge then v is leaf vertex and $v > \ell$.
    - If $\{\ell, u\}, \{\ell, v\} \in E_{\mathbf{D}}$, then $u \simeq^{\ell}_{\mathbf{D}} v$.
  - **Leaf**
- For every pair of distinct leaf vertices $u, v \in D$ it holds that $u \wedge_{\mathbf{D}} v$ is 1-type split vertex.

We denote by $\mathrm{leaf}(\mathbf{D})$ the set of all leaves of **D**.

Theorem (Laflamme-Sauer-Vuskanovic, 2006)

The big Ramsey degree of a finite graph **A** *in finite graph* **R** *is the number of diaries* **D** *such that* $\mathbf{D} \restriction_{\mathrm{leaf} D}$ *is isomorphic to* **A**.

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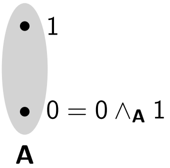

Graph daries

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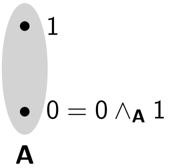

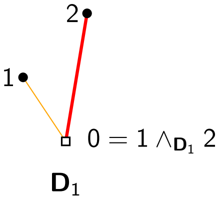

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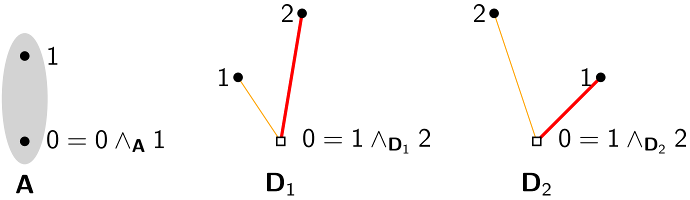

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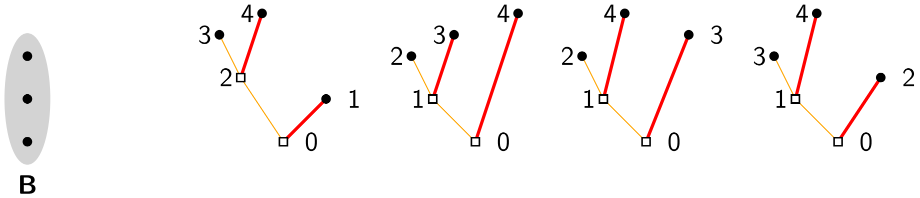

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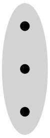
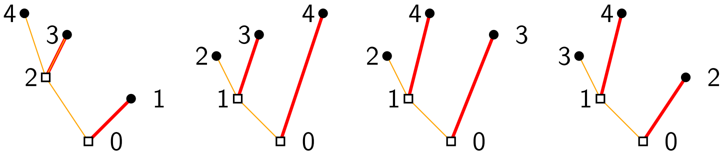


B



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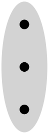

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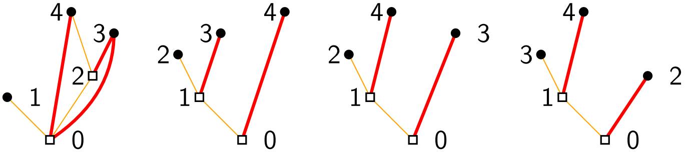

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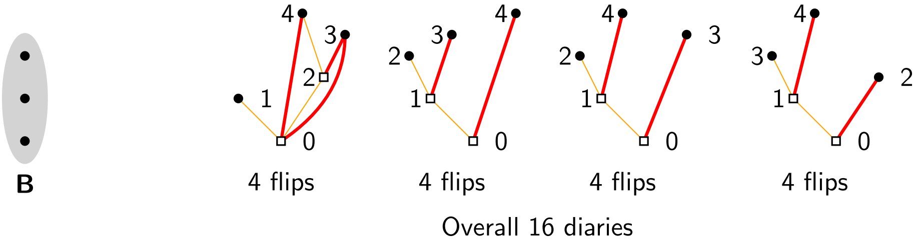

Proving optimality

Theorem (Laflamme-Sauer-Vuskanovic, 2006)

The big Ramsey degree of a finite graph **A** *in* **R** *is the number of diaries* **D** *such that* $\mathbf{D} \restriction_{\mathrm{leaf} D}$ *is isomorphic to* **A**.

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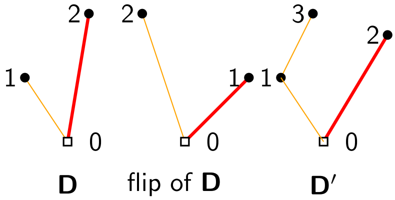
For every diary **D** *there exists a diary* $\mathbf{D}'$ *such that for every flip* $\mathbf{D}''$ *of* $\mathbf{D}'$ *has type-respecting embedding* $\mathbf{D} \to \mathbf{D}''$



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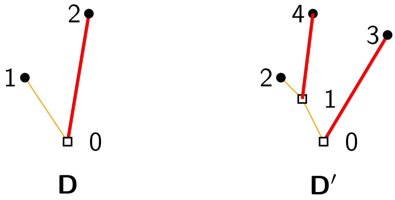

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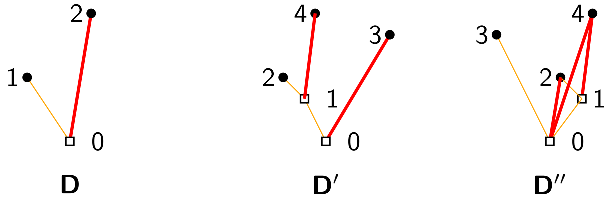

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Proof.

Let $\mathbf{E}$ be a diary of a Rado graph. Let $f : \mathbf{R} \to \mathbf{E} \restriction_{\mathrm{leaf}(\mathbf{E})}$ be an embedding given by an enemy. We prove that for every diary $\mathbf{D}$ there exits type-respecting embedding $e : \mathbf{D} \to \mathbf{E}$ such that $e[\mathrm{leaf}(\mathbf{D})] \subseteq f[R]$.

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1. Fix finite diary $\mathbf{D}$. Let $\mathbf{D}'$ be given by the flip property.
2. WLOG $\mathbf{D}' = \mathbf{R} \restriction_{|D'|}$.
3. Let g be given by the type stabilization lemma for $\mathbf{R}'$.
4. One can adjust $f \circ g$ to be embedding of flip of diary $\mathbf{D}'$ to $\mathbf{E}$.

□

Constructing embedding of diaries

Lemma (Recall: Type stabilization lemma)

Let $\mathbf{R}$ *be an enumerated Rado graph and* $\mathbf{K}$ *an enumerated graph.*
For every embedding $f : \mathbf{R} \to \mathbf{K}$ *there exists embedding* $g : \mathbf{R} \to \mathbf{R}$ *such that* $f \circ g$ *is order-preserving and for every* $\ell \in R \setminus \{0\}$, $u, v \in R \setminus \ell$

$$u \simeq^{\ell}_{\mathbf{R}} v \implies f(g(u)) \simeq^{f(g(\ell-1))+1}_{\mathbf{K}} f(g(v)).$$

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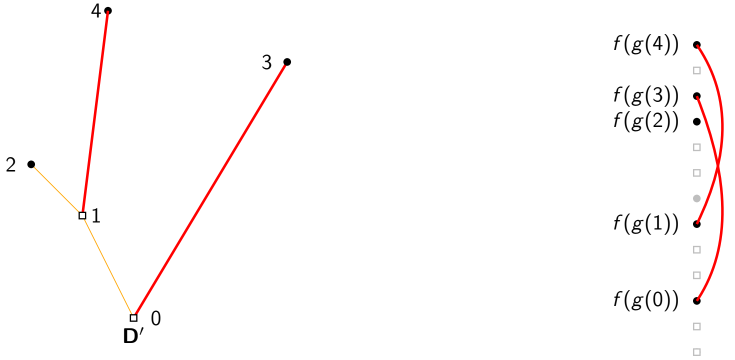

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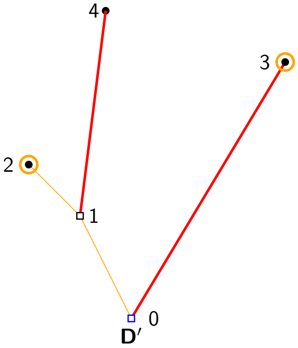

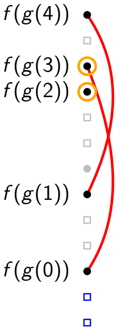

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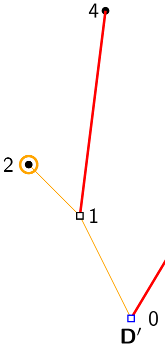

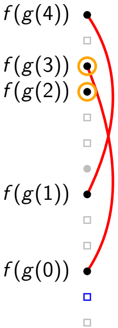

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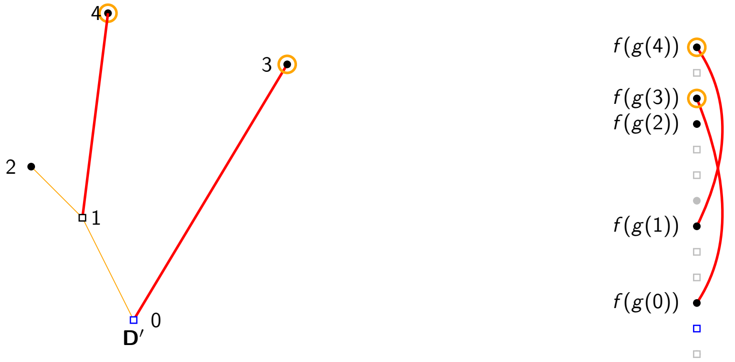

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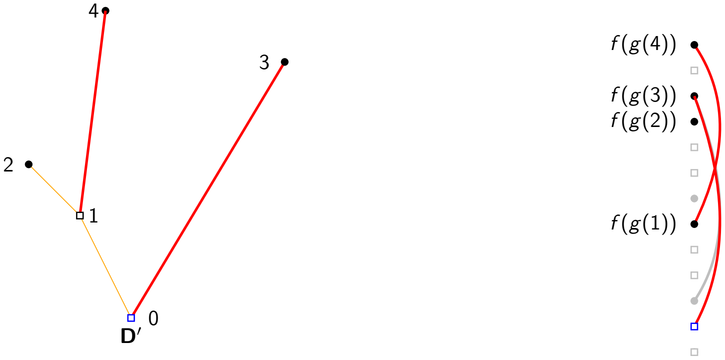

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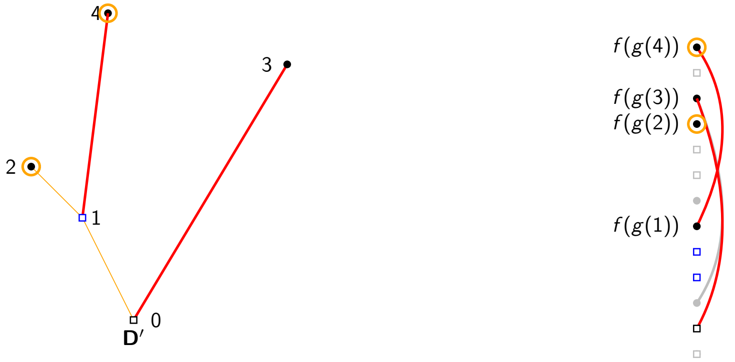

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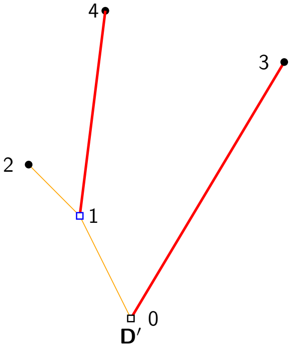


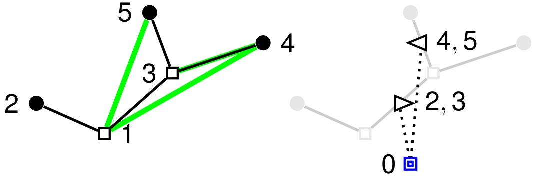
3-uniform diaries

Definition (3-uniform hypergrah diary (H., Konečný, Zucker, 2025+))

A 3-uniform hypergraph diary is an enumerated hypergraph $\mathbf{D}$ such that:

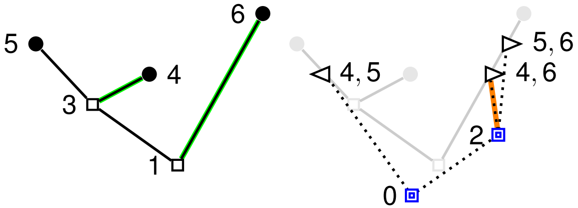
- Every vertex $\ell \in D$ has precisely one of the following roles:
  - **1-type split:**
    - There exist two leaf vertices $u, v > \ell$ such that $\ell = u \wedge_{\mathbf{D}} v$.
    - Every hyper-edge containing ℓ contains 0 and a leaf vertex $v > \ell$.
    - If $\{0, \ell, u\}, \{0, \ell, v\} \in E_{\mathbf{D}}$, then $u \simeq^{\ell}_{\mathbf{D}} v$.
  - **2-type split:**
    - $\ell = 0$ or there exitst leaves $\ell < u_0 < u_1$ and $\ell < v_1 < v_2$ such that $\ell = \{u_0, u_1\} \wedge_{\mathbf{D}} \{v_0, v_1\}$.
    - Every hyper-edge containing ℓ contains two leaf vertices $v_1 > v_0 > \ell$.
    - If $\{\ell < u_0 < u_1\}, \{\ell < v_0 < v_1\} \in E_{\mathbf{D}}$ then $\{u_0, u_1\} \simeq^{\ell}_{\mathbf{D}} \{v_0, v_1\}$.
  - **Leaf:**
    - For every $v \in \mathrm{leaf}(\mathbf{D})$, $v < \ell$ we have $\{0, v, \ell\} \in E_{\mathbf{D}} \iff v <^{\mathrm{lex}}_{\mathbf{D}} \ell$.
- The following conditions are satisfied:
  - For every $u, v \in \mathrm{leaf}(\mathbf{D})$ it holds that $u \wedge_{\mathbf{D}} v$ is 1-type split vertex.
  - For every $S, S' \in \binom{\mathrm{leaf}(\mathbf{D})}{2}$ precisely one of the following is satisfied
    - $S \wedge_{\mathbf{D}} S' = \min S = \min S'$ and $\max S \simeq^{\min S+1}_{\mathbf{D}} \max S'$,
    - $v = S \wedge_{\mathbf{D}} S'$ is 2-type split vertex. $v = 0$ iff S and S' disagrees in lexicographic order.

Diaries of non-hyperedge 1/8



$$E_{\mathbf{D}_1} = \{\{0,1,4\},\{0,1,5\},\{0,3,4\},\{0,2,4\},\{0,2,5\}\}$$

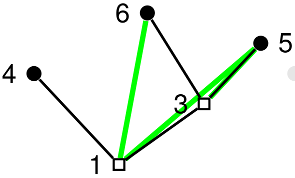
2 flips

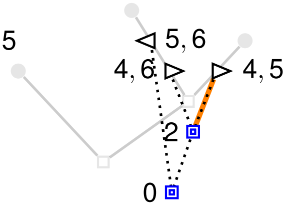


$$E_{\mathbf{D}_2} = \{\{0,1,6\},\{0,3,4\},\{0,4,6\},\{0,5,6\},\{2,4,6\}\}$$

4 flips

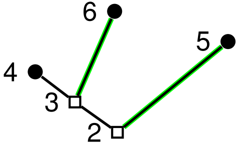
Diaries of non-hyperedge 2/8

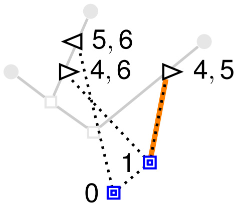




$$E_{\mathbf{D}_3} = \{\{0,1,5\},\{0,1,6\},\{0,3,5\},\{0,4,5\},\{0,4,6\},\{2,4,5\}\}$$

4 flips

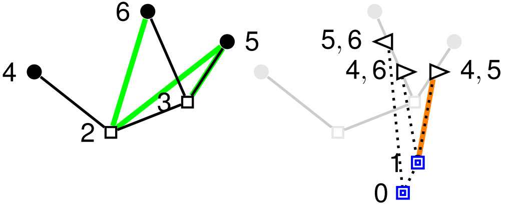




$$E_{\mathbf{D}_4} = \{\{0,2,5\},\{0,3,6\},\{0,4,5\},\{0,4,6\},\{1,4,5\}\}$$

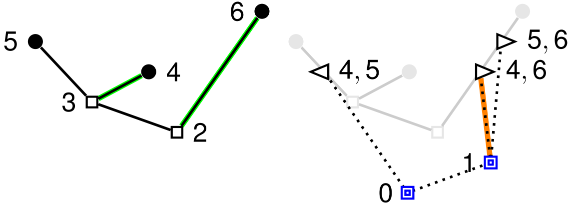
4 flips

Diaries of non-hyperedge 3/8



$$E_{\mathbf{D}_5} = \{\{0,2,5\},\{0,2,6\},\{0,3,5\},\{0,4,5\},\{0,4,6\},\{1,4,5\}\}$$

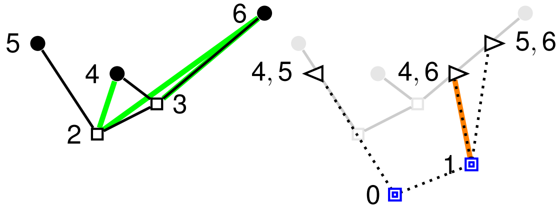
4 flips



$$E_{\mathbf{D}_6} = \{\{0,2,6\},\{0,3,4\},\{0,4,6\},\{0,5,6\},\{1,4,6\}\}$$

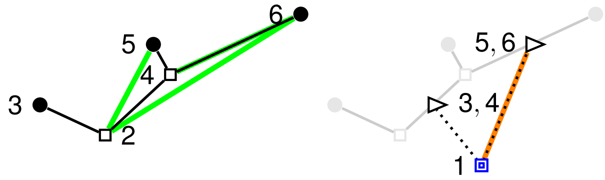
4 flips

Diaries of non-hyperedge 4/8



$$E_{\mathbf{D}_7} = \{\{0,2,4\},\{0,2,6\},\{0,3,6\},\{0,4,6\},\{0,5,6\},\{1,4,6\}\}$$

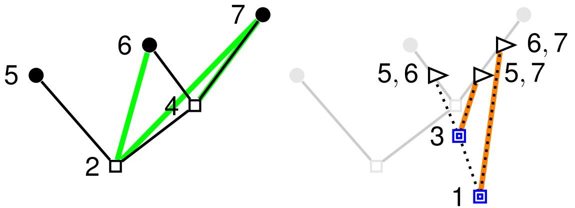
4 flips



$$E_{\mathbf{D}_8} = \{\{0,2,5\},\{0,2,6\},\{0,4,6\},\{0,3,5\},\{0,3,6\},\{0,5,6\},\{1,5,6\}\}$$

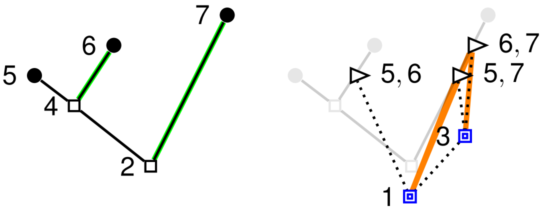
8 flips

Diaries of non-hyperedge 5/8



$$E_{\mathbf{D}_9} = \{\{0,2,6\},\{0,2,7\},\{0,4,7\},\{0,5,6\},\{0,5,7\},\{0,6,7\},\{1,6,7\},\{3,5,7\}\}$$

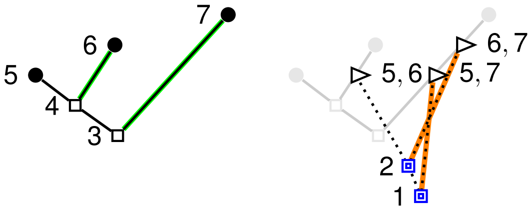
8 flips



$$E_{\mathbf{D}_{10}} = \{\{0,2,7\},\{0,4,6\},\{0,5,6\},\{0,5,7\},\{0,6,7\},\{1,5,7\},\{1,6,7\},\{3,6,7\}\}$$

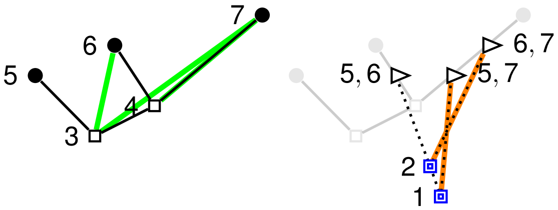
8 flips

Diaries of non-hyperedge 6/8



$$E_{\mathbf{D}_{11}} = \{\{0,3,7\},\{0,4,6\},\{0,5,6\},\{0,5,7\},\{0,6,7\},\{1,5,7\},\{2,6,7\}\}$$

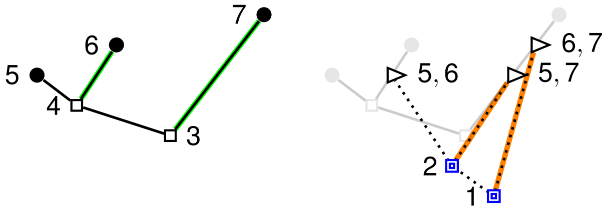
8 flips



$$E_{\mathbf{D}_{12}} = \{\{0,3,6\},\{0,3,7\},\{0,4,7\},\{0,5,6\},\{0,5,7\},\{0,6,7\},\{1,5,7\},\{2,6,7\}\}$$

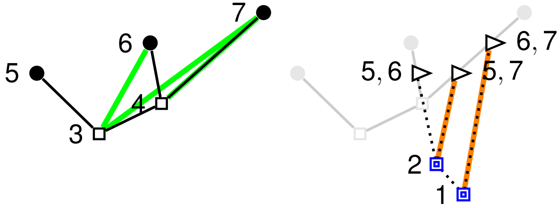
8 flips

Diaries of non-hyperedge 7/8



$$E_{\mathbf{D}_{13}} = \{\{0,3,7\},\{0,4,6\},\{0,5,6\},\{0,5,7\},\{0,6,7\},\{1,6,7\},\{2,5,7\}\}$$

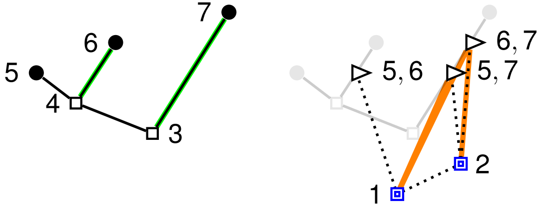
8 flips



$$E_{\mathbf{D}_{14}} = \{\{0,3,6\},\{0,3,7\},\{0,4,7\},\{0,5,6\},\{0,5,7\},\{0,6,7\},\{1,6,7\},\{2,5,7\}\}$$

8 flips

Diaries of non-hyperedge 8/8



$$E_{\mathbf{D}_{15}} = \{\{0,3,7\},\{0,4,6\},\{0,5,6\},\{0,5,7\},\{0,6,7\},\{1,5,7\},\{1,6,7\},\{2,6,7\}\}$$

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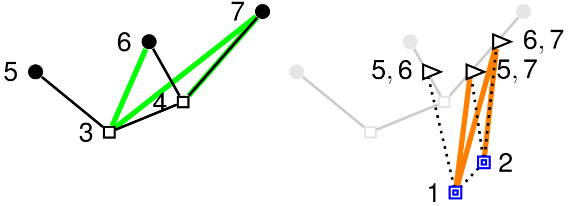

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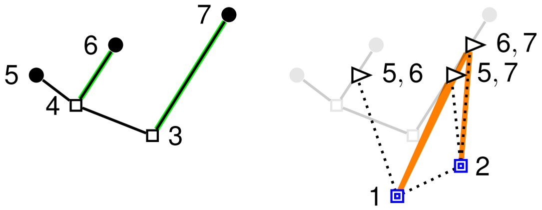

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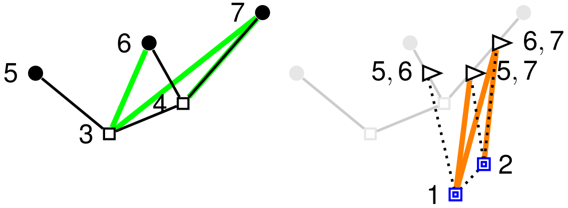

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8 flips

16 diaries of non-edge up to flips
94 diaries of non-edge overall

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2379172 diaries of 4 vertex anti-clique

16 diaries of non-edge up to flips

94 diaries of non-edge overall

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Change my mind!

16 diaries of non-edge up to flips
94 diaries of non-edge overall

2379172 diaries of 4 vertex anti-clique

Change my mind, Štěpán!

Thank you!

Winter fun in 2014

Trees with a successor operation

While most Ramsey-type theorems are concerned about regularly branching trees, we need more general notion allowing trees with finite but unbounded branching.

Definition ($\mathcal{S}$-tree)

An $\mathcal{S}$-tree is a quadruple $(T, \preceq, \Sigma, \mathcal{S})$ where $(T, \preceq)$ is a countable finitely branching tree with finitely many nodes of level 0, Σ is a set called the alphabet and $\mathcal{S}$ is a partial function $\mathcal{S}\colon T \times T^{<\omega} \times \Sigma \to T$ called the successor operation satisfying the following three axioms:

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1. If $\mathcal{S}(a, \bar{p}, c)$ is defined for some base $a \in T$, parameter $\bar{p} \in T^{<\omega}$ and character $c \in \Sigma$, then $\mathcal{S}(a, \bar{p}, c)$ is an immediate successor of a and all nodes in $\bar{p}$ have levels at most $\ell(a) - 1$.

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Example: a binary tree

Consider $\mathcal{S}$-tree is $(2^{<\omega}, \sqsubseteq, 2, \mathcal{S})$.
$\mathcal{S}$ is defined only for empty parameters $\bar{p}$ by concatenation: $\mathcal{S}(a, c) = a^\frown c$.

$$\mathcal{S}(\mathcal{S}(\mathcal{S}(\mathcal{S}(\mathcal{S}((), 0), 1), 0), 1), 1) = 01011.$$

Shape-preserving functions

Definition (Shape-preserving functions)

Let $(T, \preceq, \Sigma, \mathcal{S})$ be an $\mathcal{S}$-tree. We call an injection $F\colon T \to T$ shape-preserving if

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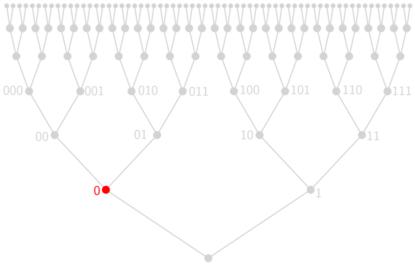

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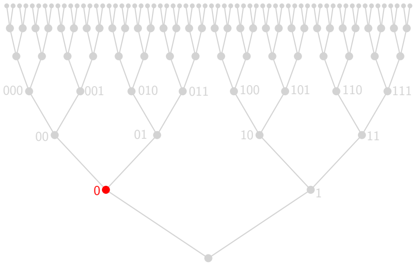

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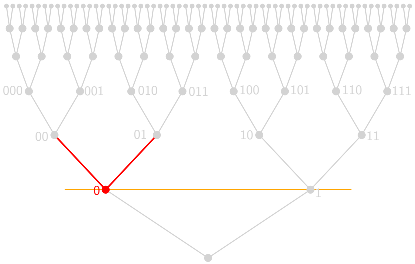

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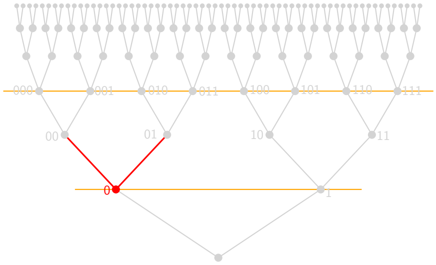

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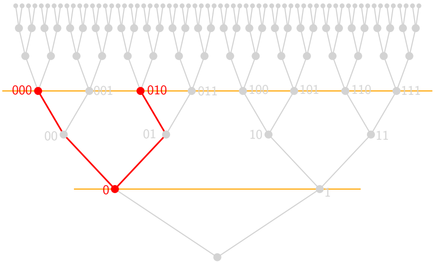

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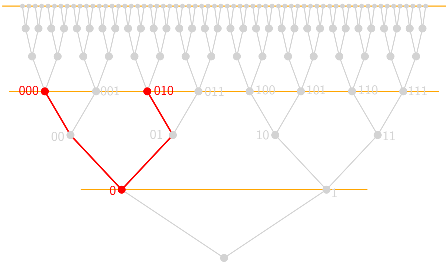

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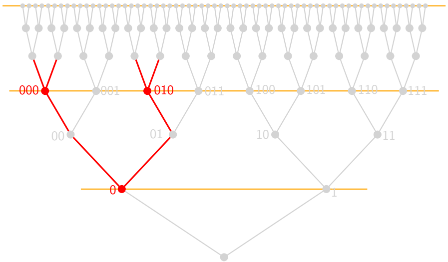

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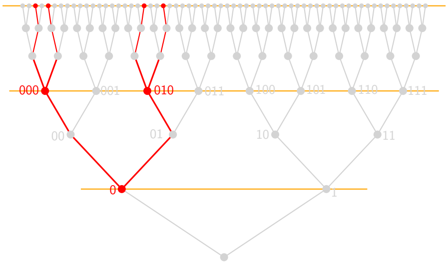

Monoids of shape-preserving functions

For a level-preserving function $F\colon S \to T$, we denote by $\tilde{F}$ the function $\tilde{F}\colon \ell(S) \to \omega$ defined by $\tilde{F}(n) = \ell(F(a))$ for some $a \in S$ with $\ell(a) = n$.
We say that F is skipping level m if $m \notin \tilde{F}[\omega]$ and that F is skipping only level m if $\tilde{F}[\omega] = \omega \setminus \{m\}$.

Definition (($\mathcal{S}, \mathcal{M}$)-tree)

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1. $\mathcal{M}$ forms a closed monoid: $\mathcal{M}$ contains the identity and is closed for compositions and limits.

Monoids of shape-preserving functions

For a level-preserving function $F\colon S \to T$, we denote by $\tilde{F}$ the function $\tilde{F}\colon \ell(S) \to \omega$ defined by $\tilde{F}(n) = \ell(F(a))$ for some $a \in S$ with $\ell(a) = n$.

We say that F is skipping level m if $m \notin \tilde{F}[\omega]$ and that F is skipping only level m if $\tilde{F}[\omega] = \omega \setminus \{m\}$.

Definition ($(\mathcal{S}, \mathcal{M})$-tree)

Given an $\mathcal{S}$-tree $(T, \preceq, \Sigma, \mathcal{S})$ and a monoid $\mathcal{M}$ of some shape-preserving functions $T \to T$, we call $(T, \preceq, \Sigma, \mathcal{S}, \mathcal{M})$ an $(\mathcal{S}, \mathcal{M})$-tree if the following three conditions are satisfied:

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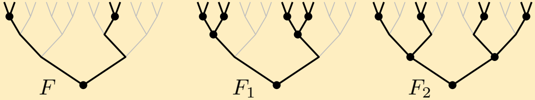

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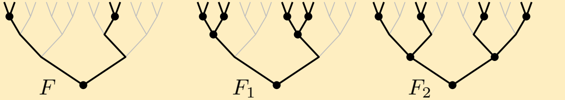
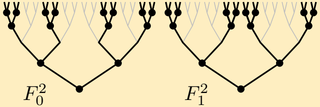


3. $\mathcal{M}$ is closed for duplication: For all n and m with $n < m \in \omega$, there exists a function $F^n_m \in \mathcal{M}$ skipping only level m such that for every $a \in T(n)$, $b \in T(m)$, $\bar{p} \in T^{<\omega}$ and $c \in \Sigma$, where $\mathcal{S}(a, \bar{p}, c)$ is defined and $\mathcal{S}(a, \bar{p}, c) \preceq b$, we have $F^n_m(b) = \mathcal{S}(b, \bar{p}, c)$.



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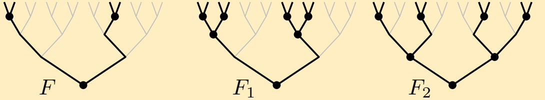

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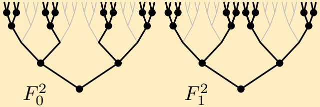

Ramsey theorem for trees with successor operation

Put $\mathcal{M}^n = \{F \in \mathcal{M} : F \restriction_{T(<n)} \text{ is identity}\}$, $\mathcal{AM}^n_k = \{F \restriction_{T(<n+k)} : F \in \mathcal{M}^n\}$.

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Let $(T, \preceq, \Sigma, \mathcal{S}, \mathcal{M})$ be an $(\mathcal{S}, \mathcal{M})$-tree. Then, for every pair $n, k \in \omega$ and every finite coloring χ of $\mathcal{AM}^n_k$, there exists $F \in \mathcal{M}^n$ such that χ is constant when restricted to $\{F \circ g : g \in \mathcal{AM}^n_k\}$.

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4. If $|\Sigma| > 1$ and $\mathcal{M}$ is generated only by duplication and “constant” functions we obtain Graham–Rothschild theorem theorem.

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