Constrained homomorphism orders

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Joint work with Jirka Fiala and Yangjing Long

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Graph homomorphisms

(Graph) homomorphism is an edge preserving mapping:

Definition

$f : V_G \to V_H$ is a homomorphism from graph $G = (V_G, E_G)$ to graph $H = (V_H, E_H)$ if

$$(x, y) \in E_G \implies (f(x), f(y)) \in E_H$$

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- We write $G \leq H$ if there exists homomorphism from G to H.
- Homomorphisms compose, identity is a homomorphism $\implies$ homomorphism induce quasi-orders on the class of all graphs as well as the class of all directed graphs.

Quasi order to partial order

Definition

- Two graphs G and H are homomorphically equivalent if $G \leq H \leq G$.
- Every equivalency class of $\leq$ has, up to isomorphism, unique minimal (in number of vertices) representative, the graph core.

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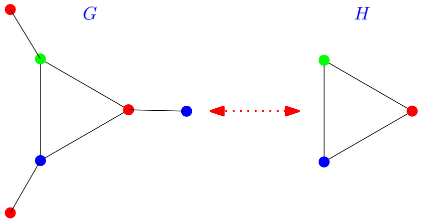
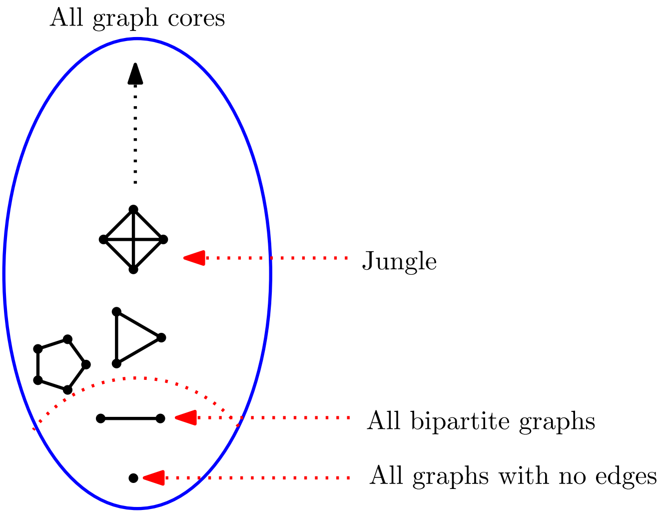


The homomorphism order



Density of homomorphism orders

Definition

Partial order is dense if for every two elements $a < b$, there is c strictly in between. $a < c < b$.

If there is no such c then (a, b) is gap.

Theorem (E. Welzl, 1982)

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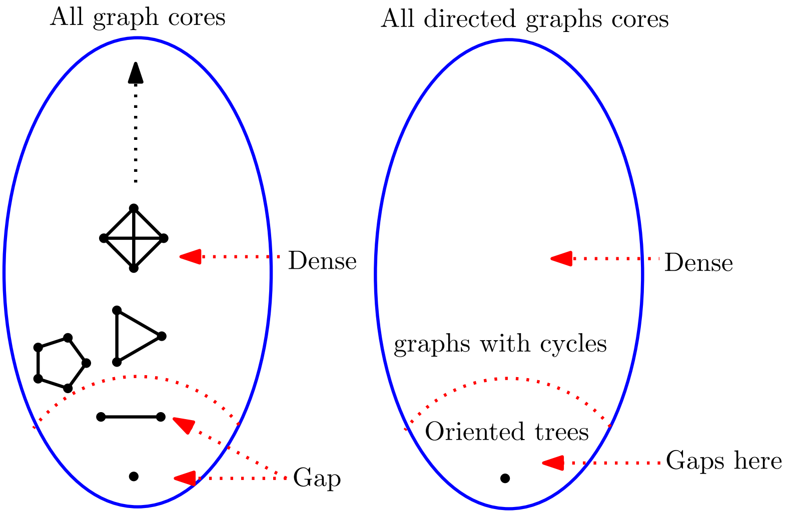
Theorem (J. Nešetřil, C. Tardiff, 1999)

The only gaps in the homomorphism order of directed graphs are pairs (G, H) *where* H *is core of orientation of a tree.*

The homomorphism order



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  - the class of all finite planar cubic graphs
  - the class of all connected series parallel graphs of girth $\geq l$
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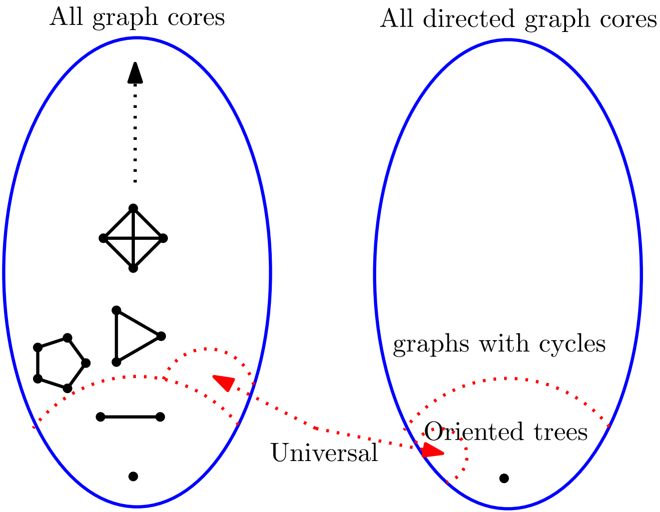
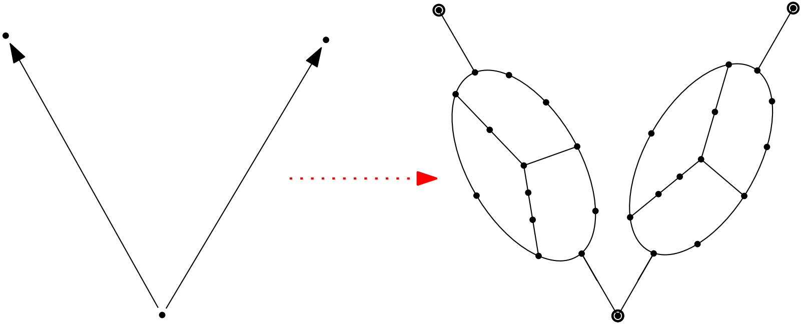


The arrow (indicator) construction

Start with class of oriented paths and transform it to new class by replacing arrows.

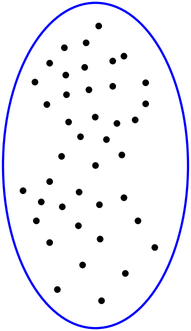


Homomorphism Dualities

Definition

Pair of directed graphs (F, D) is a simple duality pair if

$$G \not\leq D \text{ if and only if } F \leq G \text{ for all directed graphs } G.$$



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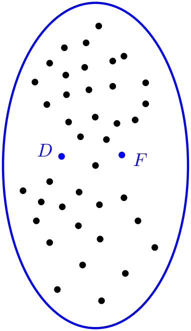

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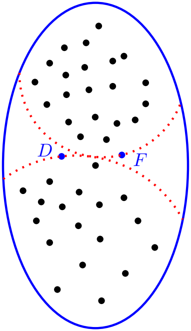

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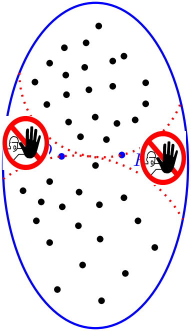

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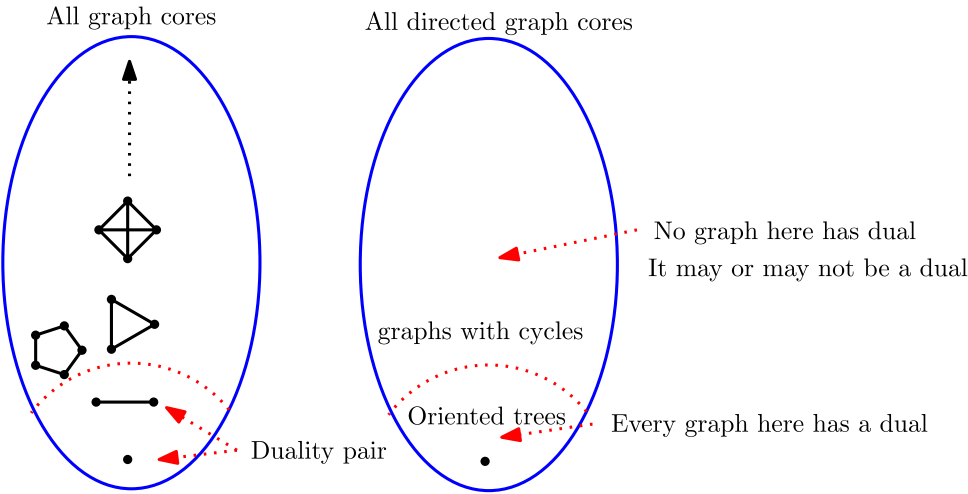
Theorem (J. Nešetřil, C. Tardiff, 1999)

There is a 1-1 correspondence between duality pairs and gaps for the class of (directed) graphs. Explicitely, given a duality pair (F, H) then $(F \times H, F)$ is a gap. Conversely, given a connected gap (G, H) then (H, G^H) is a duality pair.

Only duality pair in homomorphism order on graphs is $(K_1, \mathcal{K}_2)$.
Only orientation of trees have duals.

The homomorphism order



Constrained homomorphisms?

Homomorphism = edge preserving mapping.

- Edge preserving + injective
  $\Longrightarrow$ monomorphisms
- Edge preserving + injective + non-edge preserving
  $\Longrightarrow$ embeddings
- Edge preserving + non-edge preserving
  $\Longrightarrow$ full homomorphisms
- Edge preserving + surjective
  $\Longrightarrow$ surjective homomorphisms
- ...

Our plan

Examine individual orders induced by constrained homomorphisms and:

- Prove or disprove universality.
- Describe cores.
- Identify gaps or show density.
- Characterize duality pairs.

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In this talk: We consider locally constrained homomorphisms on connected graphs

Locally constrained homomorphisms

Denote by $N_G(u)$ the neighborhood of vertex u in G.

1. A graph homomorphism $f : G \to H$ is locally injective. if its restriction to any $N_G(u)$ and $N_H(f(u))$ is injective.
2. A graph homomorphism $f : G \to H$ is locally surjective. if its restriction to any $N_G(u)$ and $N_H(f(u))$ is surjective.
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For graphs G and H, we denote the existence of ...

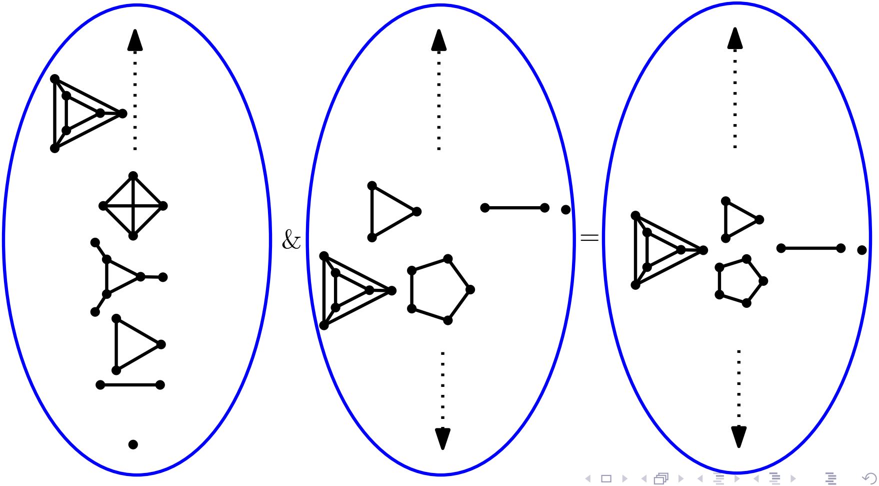
1. ... locally injective homomorphism $f : G \to H$ by $G \leq_i H$.
2. ... locally surjective homomorphism $f : G \to H$ by $G \leq_s H$.
3. ... locally bijective homomorphism $f : G \to H$ by $G \leq_b H$.

The locally constrained homomorphism orders

Connected graphs
Locally injective

Connected graphs
Locally surjective

Connected graphs
Locally bijective



Cores in locally constrained homomorphisms

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Theorem (Nešetřil 1971)

Every locally injective homomorphism $f : G \to G$ (G connected) is an automorphism of G.

Every connected graph is core in all locally constrained orders.

Degree refinement matrices

We consider connected graphs. . .

Definition

equitable partition is partitioning of vertices into classes. All vertices in a given class must have same number of neighbors in each of the classes of the partition.

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Definition

Every finite graph G admits a unique minimal equitable partition. The degree refinement matrix, $\mathrm{drm}(G)$, describes the minimal partition.
We put $G \sim^M H$ if and only if $\mathrm{drm}(G) = \mathrm{drm}(H)$.

The interplay of homomorphisms

Theorem (Kristiansen, Telle, 2000)

G, H connected graphs such that $\mathrm{drm}(G) = \mathrm{drm}(H)$. Then every locally surjective homomorphism $f : G \to H$ is locally bijective.

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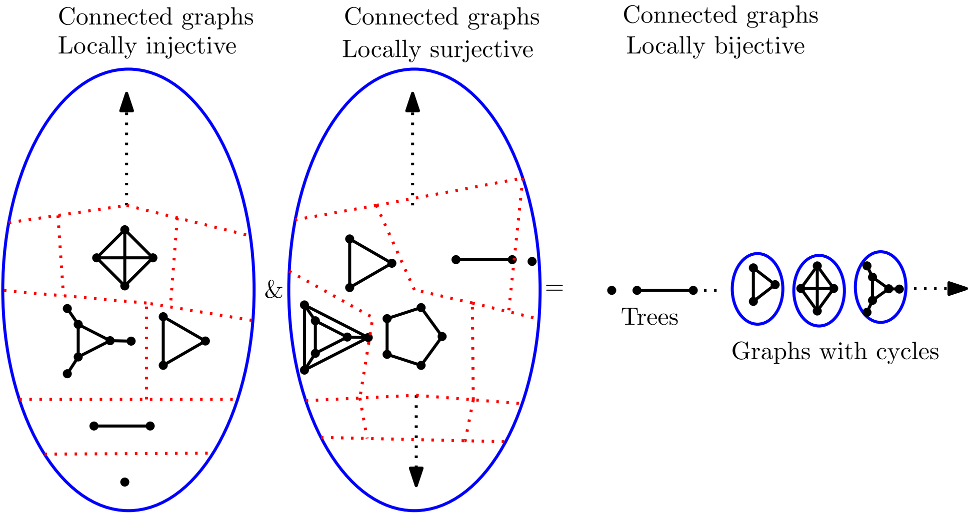
Theorem (Fiala, Maxová, 2006)

G, H are connected graphs. If $G \leq_s H$ and $G \leq_i H$ then $G \leq_b H$.

The locally constrained homomorphism orders



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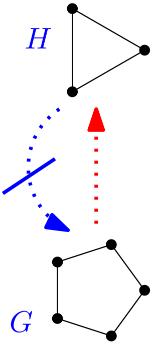

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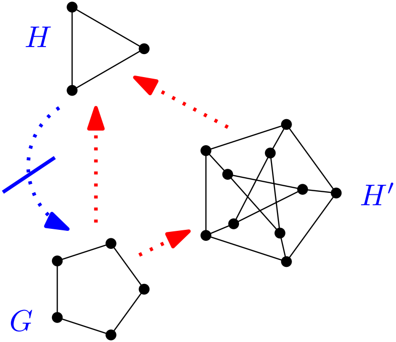

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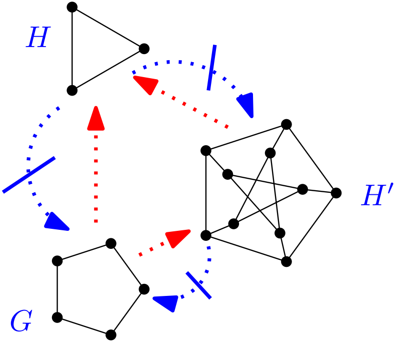

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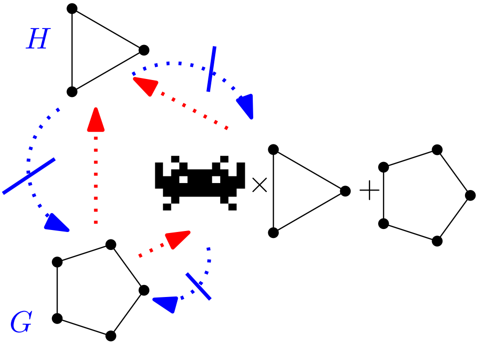

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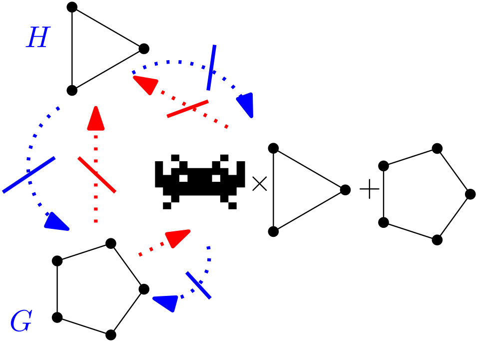

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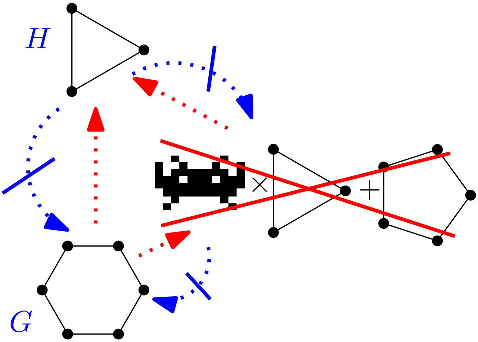

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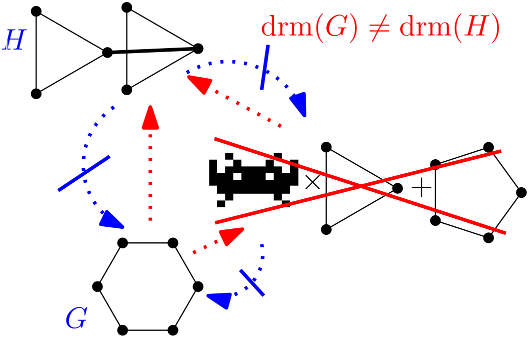

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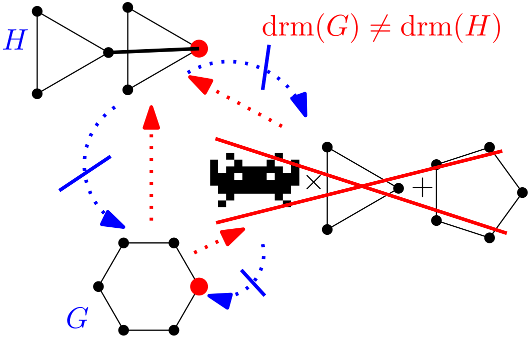

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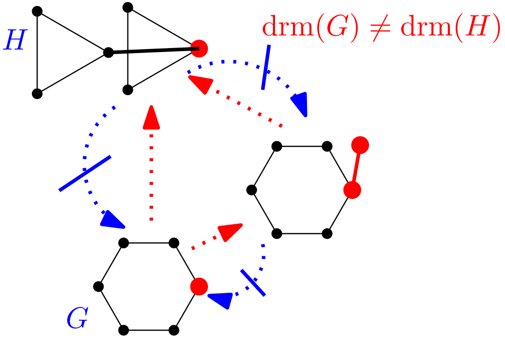

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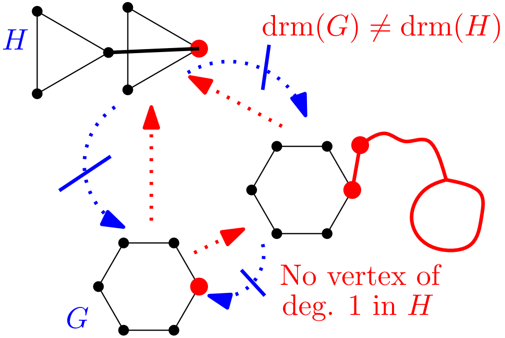

Density of locally constrained homomorphisms

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Locally surjective and bijective homomorphism orders are not dense on connected graphs.

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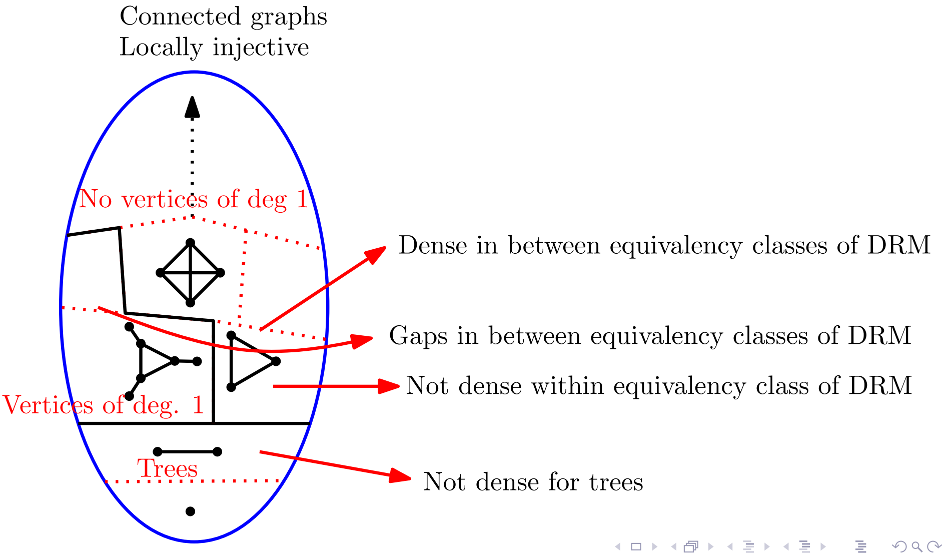
Theorem (J. Fiala, J. H., Y. Long, 2012)

Let H be any connected graph and consider locally injective homomorphisms.

(a) *There exists connected graph G such that $\mathrm{drm}(G) = \mathrm{drm}(H)$ and (G, H) is a gap if and only if H contains cycle.*
(b) *There exists connected graph G, such that $\mathrm{drm}(G) \neq \mathrm{drm}(H)$ and (G, H) is a gap if and only if H has at least one vertex of degree 1.*

The locally constrained homomorphism orders



Obvious obstacles to universality

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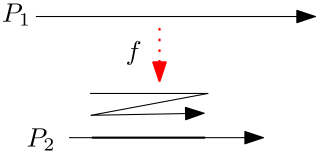
$\implies$ Locally surjective and locally bijective homomorphisms are not universal

Locally injective homomorphisms are different

Nontrivial homomorphisms of oriented paths involve folding.



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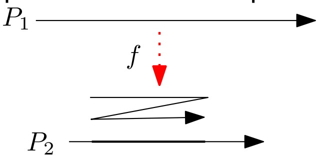
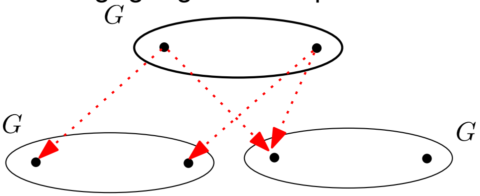


Locally injective homomorphisms never fold

$\Longrightarrow$ need for "folding" gadget G to replace vertices.



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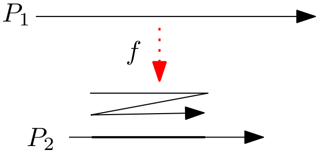

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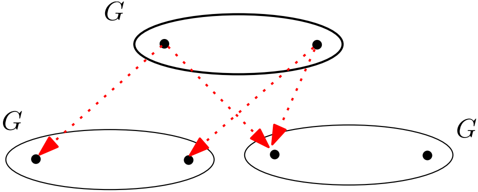

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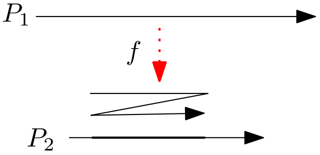

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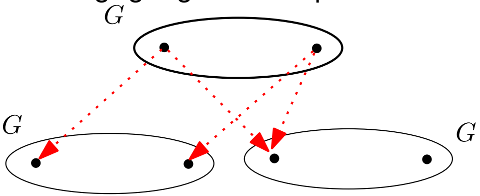

Nešetřil 1971: Every locally injective homomorphism $f : G \to G$ is an automorphism of G. $\implies$ no "folding" gadget.

Cycles are past-finite-universal

- A, B finite sets of odd primes we have $A \subseteq B$ iff

$$\prod_{b \in B} b \text{ is divisible by } \prod_{a \in A} a.$$

Partial order is past-finite if every down-set is finite.

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- C_l is oriented cycle of length l.
- $C_l \leq_i C_k$ iff l is divisible by k.

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Partial order is future-finite if every up-set is finite.

Lemma

Locally injective homomorphism order on the class of all cycles is future-finite-universal.

Representation of universal poset

- $\mathcal{P}$ is the class of finite sets of finite sets of odd primes
- For $A, B \in \mathcal{P}$ we put $A \leq^{\mathrm{dom}} B$ iff:

for every $a \in A$ there exists $b \in B$ such that $a \supseteq b$.

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- $\mathcal{P}$ is the class of finite sets of finite sets of odd primes
- For $A, B \in \mathcal{P}$ we put $A \leq^{\mathrm{dom}} B$ iff:

  for every $a \in A$ there exists $b \in B$ such that $a \supseteq b$.

Lemma

$(\mathcal{P}, \leq^{\mathrm{dom}})$ *is universal.*

Universality

Theorem (J. Fiala, J. H., Y. Long, 2012)

Locally injective homomorphisms are universal on the class of disjoint unions of cycles.

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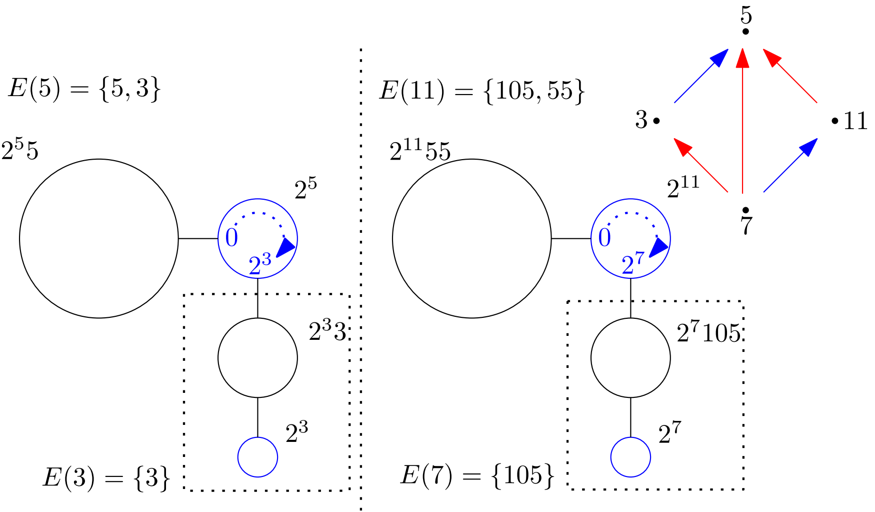
Theorem (J. Fiala, J. H., Y. Long, 2012)

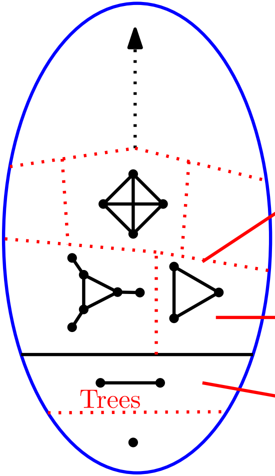
Locally injective homomorphisms are universal on the class of connected graphs.

Fractal like structure



The locally constrained homomorphism orders

Connected graphs
Locally injective



Universal across DRM classes

Future finite universal wihin every eq. class of DRM

Past finite universal on class of trees
(conincide with the embedding order)

Dualities in locally constrained homomorphisms

Definition

Pair of finite sets of directed graphs $(\mathcal{F}, \mathcal{D})$ is a generalized finite duality pair if for for every directed graph G there exists $F \in \mathcal{F}$ such that $F \to G$ if and only if $G \to D$ for no $D \in \mathcal{D}$.

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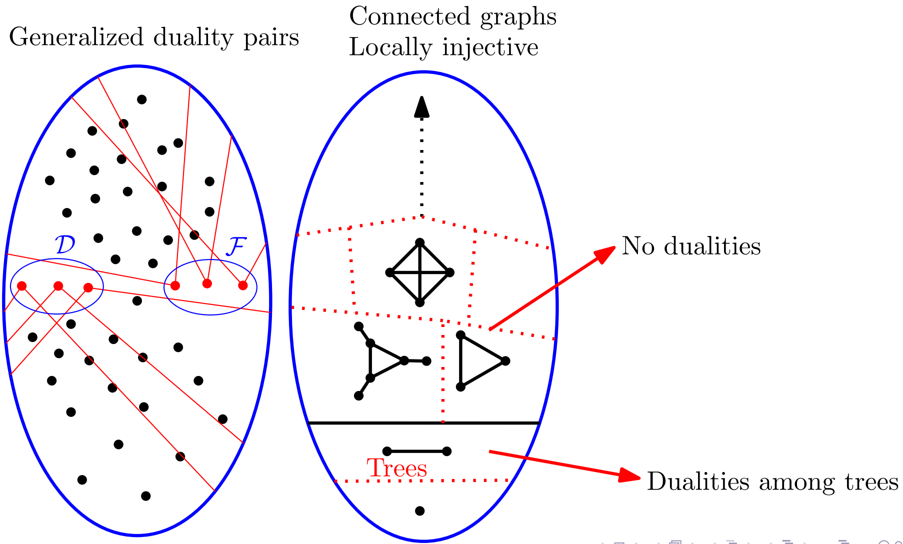
- *No generalized finite dualities for locally surjective homomorphisms.*

The locally constrained homomorphism orders



Summary I

- Trivial cores:
  monomorphisms, embeddings, surjective homomorphisms, locally constrained homomorphisms on connected graphs (Nešetřil).
- Nontrivial cores (polynomial time decidable):
  full homomorphisms (thin graphs), locally constrained homomorphisms, surjective multihomomorphisms, partial surjective multihomomorphisms.
- Nontrivial cores (NP-complete):
  homomorphisms (Hell, Nešetřil).

Summary II

- Past-finite-universal:
  monomorphisms, embeddings, full homomorphisms.
- Future-finite-universal:
  surjective homomorphisms, (partial) surjective multihomomorphisms, locally surjective/bijective on connected graphs.
- Universal:
  homomorphisms (Pultr, Trnková; JH, Nešetřil), locally constrained, locally injective on connected graphs.

Summary III

Definition

We say that a pair $(\mathcal{F}, \mathcal{D})$ of finite sets of graphs is a generalized duality pair if for all directed graphs G

$$\forall_{D \in \mathcal{D}} G \not\leq D \text{ if and only if } \exists_{F \in \mathcal{F}} F \leq G$$

- all left generalized duals:
  monomorphisms, embeddings, full homomorphisms (Feder, Hell; Ball, Nešetřil, Pultr; Xie), locally constrained homomorphisms.
- all right generalized duals:
  surjective homomorphisms, surjective multihomomorphisms.
- some duals (characterized)
  homomorphisms (Nešetřil, Tardif).

Questions?

Thank you. . .