Constrained homomorphism orders

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Joint work with Jirka Fiala and Yangjing Long

BGW2012, Bordeaux
Graph homomorphisms

(Graph) homomorphism is an edge preserving mapping:

**Definition**

$f : V_G \rightarrow V_H$ is a homomorphism from graph $G = (V_G, E_G)$ to graph $H = (V_H, E_H)$ if

$$(x, y) \in E_G \implies (f(x), f(y)) \in E_H$$
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**Graph homomorphisms**

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Constrained homomorphism orders
Graph homomorphisms

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- We write $G \leq H$ if there exists homomorphism from $G$ to $H$.
- Homomorphisms compose, identity is a homomorphism $\implies$ homomorphism induce quasi-orders on the class of all graphs as well as the class of all directed graphs.
Quasi order to partial order

Definition

Two graphs $G$ and $H$ are homomorphically equivalent if $G \leq H \leq G$.

Every equivalency class of $\leq$ has, up to isomorphism, unique minimal (in number of vertices) representative, the graph core.
Quasi order to partial order

Definition

- Two graphs $G$ and $H$ are **homomorphically equivalent** if $G \leq H \leq G$.

- Every equivalency class of $\leq$ has, up to isomorphism, unique minimal (in number of vertices) representative, the **graph core**.
The homomorphism order

All graph cores

Jungle

All bipartite graphs

All graphs with no edges

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Constrained homomorphism orders
Density of homomorphism orders

Definition

Partial order is **dense** if for every two elements \(a < b\), there is \(c\) strictly in between. \(a < c < b\).

If there is no such \(c\) then \((a, b)\) is **gap**.

Theorem (E. Welzl, 1982)

*The only gap in the homomorphism order of graphs is \((K_1, K_2)\)*
Definition
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**Theorem (E. Welzl, 1982)**

*The only gap in the homomorphism order of graphs is $(K_1, K_2)$*

**Theorem (J. Nešetřil, C. Tardiff, 1999)**

*The only gaps in the homomorphism order of directed graphs are pairs $(G, H)$ where $H$ is core of orientation of a tree.*
The homomorphism order

All graph cores

- Dense
- Gap

All directed graphs cores

- Dense
- Oriented trees
- Gaps here
- Graphs with cycles

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Constrained homomorphism orders
Universal partial order

Definition

Countable partial order is universal if it contains every countable partial order as an suborder.
Universal partial order

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- Graph homomorphisms are universal in categorical sense (Pultr, Trnková, 1980)
- Homomorphism **order** remain universal on the class of oriented paths (H., Nešetřil, 2003)
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  \[\implies\] homomorphism order is universal on following classes
  - the class of all finite planar cubic graphs
  - the class of all connected series parallel graphs of girth \( \geq l \)
  - \( \ldots \)
- Homomorphism order is universal on partial orders and lattices (Lehtonen, 2008)
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The homomorphism order

All graph cores

All directed graph cores

Oriented trees

graphs with cycles

Universal

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Constrained homomorphism orders
The arrow (indicator) construction

Start with class of oriented paths and transform it to new class by replacing arrows.
Homomorphism Dualities

Definition

Pair of directed graphs \((F, D)\) is a **simple duality pair** if

\[ G \nleq D \text{ if and only if } F \nleq G \text{ for all directed graphs } G. \]
Definition

Pair of directed graphs \((F, D)\) is a **simple duality pair** if

\[ G \not\leq D \text{ if and only if } F \leq G \text{ for all directed graphs } G. \]
Definition

Pair of directed graphs \((F, D)\) is a **simple duality pair** if

\[ G \not\cong D \text{ if and only if } F \leq G \text{ for all directed graphs } G. \]
**Homomorphism Dualities**

**Definition**

Pair of directed graphs \((F, D)\) is a **simple duality pair** if

\[ G \not\cong D \text{ if and only if } F \leq G \text{ for all directed graphs } G. \]
Homomorphism Dualities

**Definition**

Pair of directed graphs $(F, D)$ is a simple duality pair if

$$G \not\preceq D \iff F \preceq G$$

for all directed graphs $G$.

**Theorem (J. Nešetřil, C. Tardiff, 1999)**

There is a 1-1 correspondence between duality pairs and gaps for the class of (directed) graphs. Explicitly, given a duality pair $(F, H)$ then $(F \times H, F)$ is a gap. Conversely, given a connected gap $(G, H)$ then $(H, G^H)$ is a duality pair.

Only duality pair in homomorphism order on graphs is $(K_1, K_2)$. Only orientation of trees have duals.
The homomorphism order

All graph cores

Oriented trees

graphs with cycles

Duality pair

All directed graph cores

Every graph here has a dual

No graph here has dual

It may or may not be a dual

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Constrained homomorphism orders
Constrained homomorphisms?

Homomorphism = edge preserving mapping.

- Edge preserving + injective $\implies$ monomorphisms
- Edge preserving + injective + non-edge preserving $\implies$ embeddings
- Edge preserving + non-edge preserving $\implies$ full homomorphisms
- Edge preserving + surjective $\implies$ surjective homomorphisms
- ...
Our plan

Examine individual orders induced by constrained homomorphisms and:

- Prove or disprove universality.
- Describe cores.
- Identify gaps or show density.
- Characterize duality pairs.
Our plan

Examine individual orders induced by constrained homomorphisms and:

- Prove or disprove universality.
- Describe cores.
- Identify gaps or show density.
- Characterize duality pairs.

In this talk: We consider locally constrained homomorphisms on connected graphs
Locally constrained homomorphisms

Denote by $N_G(u)$ the neighborhood of vertex $u$ in $G$.

1. A graph homomorphism $f : G \to H$ is locally injective. if its restriction to any $N_G(u)$ and $N_H(f(u))$ is injective.

2. A graph homomorphism $f : G \to H$ is locally surjective. if its restriction to any $N_G(u)$ and $N_H(f(u))$ is surjective.

3. A graph homomorphism $f : G \to H$ is locally bijective. if its restriction to any $N_G(u)$ and $N_H(f(u))$ is bijective.
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For graphs $G$ and $H$, we denote the existence of . . .

1. . . . locally injective homomorphism $f : G \to H$ by $G \leq_i H$.

2. . . . locally surjective homomorphism $f : G \to H$ by $G \leq_s H$.

3. . . . locally bijective homomorphism $f : G \to H$ by $G \leq_b H$. 
The locally constrained homomorphism orders

Connected graphs
Locally injective

&

Connected graphs
Locally surjective

=

Connected graphs
Locally bijective
Cores in locally constrained homomorphisms

Lemma

*Every locally surjective/bijective homomorphisms* $F : G \rightarrow H$ *is surjective when* $H$ *is connected.*
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$G \leq_s H \leq_s G$
Cores in locally constrained homomorphisms

Lemma

*Every locally surjective/bijective homomorphisms* $F : G \rightarrow H$ *is surjective when* $H$ *is connected.*

$$G \leq_s H \leq_s G \implies |G| \geq |H| \geq |G| \implies |G| = |H|$$
Cores in locally constrained homomorphisms

Lemma

Every locally surjective/bijective homomorphisms $F : G \rightarrow H$ is surjective when $H$ is connected.

$G \leq_s H \leq_s G \implies |G| \geq |H| \geq |G| \implies |G| = |H|$

$\implies$ no interesting cores for locally injective/surjective homomorphisms on connected graphs
The homomorphism order
Constrained homomorphisms
Locally constrained homomorphism orders
Density
Universality
Dualities

Cores in locally constrained homomorphisms

Lemma

Every locally surjective/bijective homomorphisms $F : G \to H$ is surjective when $H$ is connected.

$G \leq_{s} H \leq_{s} G \implies |G| \geq |H| \geq |G| \implies |G| = |H|$

$\implies$ no interesting cores for locally injective/surjective homomorphisms on connected graphs

Theorem (Nešetřil 1971)

Every locally injective homomorphism $f : G \to G$ (G connected) is an automorphism of G.

Every connected graph is core in all locally constrained orders.
We consider connected graphs.

**Definition**

*equitable partition* is partitioning of vertices into classes. All vertices in a given class must have same number of neighbors in each of the classes of the partition.
Degree refinement matrices

We consider connected graphs...

Definition

**equitable partition** is partitioning of vertices into classes. All vertices in a given class must have same number of neighbors in each of the classes of the partition.

**Degree matrix** describe the sizes of neighborhoods in a given equitable partition (in a canonical way).
Degree refinement matrices

We consider connected graphs...

**Definition**

**equitable partition** is partitioning of vertices into classes. All vertices in a given class must have same number of neighbors in each of the classes of the partition. **Degree matrix** describe the sizes of neighborhoods in a given equitable partition (in a canonical way).

**Definition**

Every finite graph $G$ admits a unique minimal equitable partition. The **degree refinement matrix**, $\text{drm}(G)$, describes the minimal partition.
We consider connected graphs. . .

**Definition**

An **equitable partition** is partitioning of vertices into classes. All vertices in a given class must have the same number of neighbors in each of the classes of the partition. A **degree matrix** describes the sizes of neighborhoods in a given equitable partition (in a canonical way).

**Definition**

Every finite graph $G$ admits a unique minimal equitable partition. The **degree refinement matrix**, $\text{drm}(G)$, describes the minimal partition. We put $G \sim^M H$ if and only if $\text{drm}(G) = \text{drm}(H)$.
The interplay of homomorphisms

Theorem (Kristiansen, Telle, 2000)

G, H connected graphs such that $\text{drm}(G) = \text{drm}(H)$. Then every locally surjective homomorphism $f : G \to H$ is locally bijective.
The interplay of homomorphisms

Theorem (Kristiansen, Telle, 2000)

*G, H connected graphs such that* \( \text{drm}(G) = \text{drm}(H) \). *Then every locally surjective homomorphism* \( f : G \rightarrow H \) *is locally bijective.*

Theorem (Fiala, Kratochvíl, 2001)

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The interplay of homomorphisms

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$G, H$ connected graphs such that $\text{drm}(G) = \text{drm}(H)$. Then every locally injective homomorphism $f : G \to H$ is locally bijective.

Theorem (Fiala, Maxová, 2006)

$G, H$ are connected graphs. If $G \leq_s H$ and $G \leq_i H$ then $G \leq_b H$. 
The locally constrained homomorphism orders

Connected graphs
Locally injective

Connected graphs
Locally surjective

Connected graphs
Locally bijective

Trees

Graphs with cycles

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Density

\[ G \leq_s H \implies |G| \leq |H| \implies \text{Locally surjective/bijective homomorphism order is not dense} \]

Locally injective homomorphisms?
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Locally injective homomorphisms?
Density

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The homomorphism order
Constrained homomorphisms
Locally constrained homomorphism orders

Density

\( G \leq_{s} H \implies |G| \leq |H| \implies \) Locally surjective/bijective homomorphism order is not dense

Locally injective homomorphisms?

\[ \text{drm}(G) \neq \text{drm}(H) \]

No vertex of deg. 1 in \( H \)
Density of locally constrained homomorphisms

Theorem (J. Fiala, J. H., Y. Long, 2012)

*Locally surjective and bijective homomorphism orders are not dense on connected graphs.*
Density of locally constrained homomorphisms

**Theorem (J. Fiala, J. H., Y. Long, 2012)**

Locally surjective and bijective homomorphism orders are not dense on connected graphs.

**Theorem (J. Fiala, J. H., Y. Long, 2012)**

Let $H$ be any connected graph and consider locally injective homomorphisms.

(a) There exists connected graph $G$ such that $\text{drm}(G) = \text{drm}(H)$ and $(G, H)$ is a gap if and only if $H$ contains cycle.

(b) There exists connected graph $G$, such that $\text{drm}(G) \neq \text{drm}(H)$ and $(G, H)$ is a gap if and only if $H$ has at least one vertex of degree 1.
The locally constrained homomorphism orders

Connected graphs
Locally injective

- No vertices of deg 1
- Vertices of deg. 1
- Trees

- Dense in between equivalency classes of DRM
- Gaps in between equivalency classes of DRM
- Not dense within equivalency class of DRM
- Not dense for trees
Obvious obstacles to universality

\[ G \preceq_s H \iff |G| \geq_s |H| \]
Obvious obstacles to universality

\[ G \leq_s H \implies |G| \geq_s |H| \]

\[ \implies \text{no infinite decreasing chains} \]
Obvious obstacles to universality

\[ G \leq_s H \implies |G| \geq_s |H| \]

\implies \text{no infinite decreasing chains}

\implies \text{Locally surjective and locally bijective homomorphisms are not universal}
Locally injective homomorphisms are different

Nontrivial homomorphisms of oriented paths involve folding.

\[ P_1 \xrightarrow{f} P_2 \]
Locally injective homomorphisms are different

Nontrivial homomorphisms of oriented paths involve folding.

Locally injective homomorphisms never fold
\[\Rightarrow\text{ need for “folding” gadget } G \text{ to replace vertices.}\]
Locally injective homomorphisms are different

Nontrivial homomorphisms of oriented paths involve folding.

Locally injective homomorphisms never fold

⇒ need for “folding” gadget $G$ to replace vertices.

Nešetřil 1971: Every locally injective homomorphism $f : G \rightarrow G$ is an automorphism of $G$. 
Locally injective homomorphisms are different

Nontrivial homomorphisms of oriented paths involve folding.

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\[ \implies \text{need for “folding” gadget } G \text{ to replace vertices.} \]

Nešetřil 1971: Every locally injective homomorphism \( f : G \rightarrow G \)
is an automorphism of \( G \). \[ \implies \text{no “folding” gadget.} \]
Cycles are past-finite-universal

- \( A, B \) finite sets of odd primes we have \( A \subseteq B \) iff
  \[
  \prod_{b \in B} b \text{ is divisible by } \prod_{a \in A} a.
  \]

Partial order is past-finite if every down-set is finite.

Lemma

The divisibility order is past-finite-universal.
Cycles are past-finite-universal

- $A, B$ finite sets of odd primes we have $A \subseteq B$ iff
  $$\prod_{b \in B} b \text{ is divisible by } \prod_{a \in A} a.$$  

Partial order is past-finite if every down-set is finite.

**Lemma**

*The divisibility order is past-finite-universal.*

- $C_l$ is oriented cycle of length $l$.
- $C_l \leq_i C_k$ iff $l$ is divisible by $k$. 
Cycles are past-finite-universal

- A, B finite sets of odd primes we have \( A \subseteq B \) iff \( \prod_{b \in B} b \) is divisible by \( \prod_{a \in A} a \).

Partial order is past-finite if every down-set is finite.

**Lemma**

The divisibility order is past-finite-universal.

- \( C_l \) is oriented cycle of length \( l \).
- \( C_l \leq_i C_k \) iff \( l \) is divisible by \( k \).

Partial order is future-finite if every up-set is finite.

**Lemma**

Locally injective homomorphism order on the class of all cycles is future-finite-universal.
$\mathcal{P}$ is the class of finite sets of finite sets of odd primes

For $A, B \in \mathcal{P}$ we put $A \leq^{\text{dom}} B$ iff:

for every $a \in A$ there exists $b \in B$ such that $a \supseteq b$. 
$\mathcal{P}$ is the class of finite sets of finite sets of odd primes

For $A, B \in \mathcal{P}$ we put $A \leq_{\text{dom}} B$ iff:

for every $a \in A$ there exists $b \in B$ such that $a \supseteq b$.

**Lemma**

$(\mathcal{P}, \leq_{\text{dom}})$ is universal.
Theorem (J. Fiala, J. H., Y. Long, 2012)

*Locally injective homomorphisms are universal on the class of disjoint unions of cycles.*
Theorem (J. Fiala, J. H., Y. Long, 2012)

Locally injective homomorphisms are universal on the class of disjoint unions of cycles.

Corollary (J. Fiala, J. H., Y. Long, 2012)

Homomorphism order is universal on the class of disjoint unions of cycles oriented clockwise.
Theorem (J. Fiala, J. H., Y. Long, 2012)

*Locally injective homomorphisms are universal on the class of disjoint unions of cycles.*

Corollary (J. Fiala, J. H., Y. Long, 2012)

*Homomorphism order is universal on the class of disjoint unions of cycles oriented clockwise.*

Theorem (J. Fiala, J. H., Y. Long, 2012)

*Locally injective homomorphisms are universal on the class of connected graphs.*
Fractal like structure

$E(5) = \{5, 3\}$

$E(3) = \{3\}$

$E(7) = \{105\}$

$E(11) = \{105, 55\}$
The locally constrained homomorphism orders

- Connected graphs
- Locally injective
- Trees

Past finite universal on class of trees (coincide with the embedding order)
Future finite universal within every eq. class of DRM
Universal across DRM classes
Dualities in locally constrained homomorphisms

Definition

Pair of finite sets of directed graphs \((F, D)\) is a **generalized finite duality pair** if for every directed graph \(G\) there exists \(F \in F\) such that \(F \rightarrow G\) if and only if \(G \rightarrow D\) for no \(D \in D\).
Dualities in locally constrained homomorphisms

Definition

Pair of finite sets of directed graphs \((\mathcal{F}, \mathcal{D})\) is a **generalized finite duality pair** if for every directed graph \(G\) there exists \(F \in \mathcal{F}\) such that \(F \rightarrow G\) if and only if \(G \rightarrow D\) for no \(D \in \mathcal{D}\).

Theorem

\(\mathcal{D}\) have left dual in locally injective homomorphism order if and only if \(\mathcal{D}\) consists of finite family or trees.
Defining a generalized finite duality pair:

A pair of finite sets of directed graphs \((\mathcal{F}, \mathcal{D})\) is a generalized finite duality pair if for every directed graph \(G\) there exists \(F \in \mathcal{F}\) such that \(F \rightarrow G\) if and only if \(G \rightarrow D\) for no \(D \in \mathcal{D}\).

**Theorem:**

- \(\mathcal{D}\) have left dual in locally injective homomorphism order if and only if \(\mathcal{D}\) consists of finite family or trees.
- No generalized finite dualities for locally bijective homomorphisms.
Dualities in locally constrained homomorphisms

Definition
Pair of finite sets of directed graphs $(\mathcal{F}, \mathcal{D})$ is a **generalized finite duality pair** if for every directed graph $G$ there exists $F \in \mathcal{F}$ such that $F \rightarrow G$ if and only if $G \rightarrow D$ for no $D \in \mathcal{D}$.

Theorem
- $\mathcal{D}$ have left dual in locally injective homomorphism order if and only if $\mathcal{D}$ consists of finite family or trees.
- No generalized finite dualities for locally bijective homomorphisms.
- No generalized finite dualities for locally surjective homomorphisms.
The locally constrained homomorphism orders

Generalized duality pairs

Connected graphs
Locally injective

No dualities

Dualities among trees
Summary 1

- **Trivial cores:** monomorphisms, embeddings, surjective homomorphisms, locally constrained homomorphisms on connected graphs (Nešetřil).

- **Nontrivial cores (polynomial time decidable):** full homomorphisms (thin graphs), locally constrained homomorphisms, surjective multihomomorphisms, partial surjective multihomomorphisms.

- **Nontrivial cores (NP-complete):** homomorphisms (Hell, Nešetřil).
Summary II

- **Past-finite-universal:**
  monomorphisms, embeddings, full homomorphisms.

- **Future-finite-universal:**
  surjective homomorphisms, (partial) surjective multihomomorphisms, locally surjective/bijective on connected graphs.

- **Universal:**
  homomorphisms (Pultr, Trnková; JH, Nešetřil), locally constrained, locally injective on connected graphs.
Definition

We say that a pair \((\mathcal{F}, \mathcal{D})\) of finite sets of graphs is a **generalized duality pair** if for all directed graphs \(G\)

\[
\forall_{D \in \mathcal{D}} \exists F \in \mathcal{F} \quad \exists ! F \leq G
\]

- **all left generalized duals:**
  - monomorphisms, embeddings, full homomorphisms (Feder, Hell; Ball, Nešetřil, Pultr; Xie), locally constrained homomorphisms.
- **all right generalized duals:**
  - surjective homomorphisms, surjective multihomomorphisms.
- **some duals (characterized)**
  - homomorphisms (Nešetřil, Tardif).
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Questions?

Thank you...