Big Ramsey degrees of homogeneous structures

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Big Ramsey Degrees

Theorem (Infinite Ramsey Theorem, 1930)

$$\forall_{p,k\geq 1} : \omega \longrightarrow (\omega)^{p}_{k,1}.$$

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$N \longrightarrow (n)^p_{k,t}$: For every partition of $\binom{\omega}{p}$ into k classes (colours) there exists $X \in \binom{\omega}{\omega}$ such that $\binom{X}{p}$ belongs to at most t parts.

($t = 1$ means that $\binom{X}{p}$ is monochromatic.)

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Let $\mathcal{O}$ be the class of all finite linear orders.

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$\binom{\mathbf{B}}{\mathbf{A}}$ is the set of all embeddings of structure **A** to structure **B**.

$\mathbf{C} \longrightarrow (\mathbf{B})^{\mathbf{A}}_{k,t}$: For every k-colouring of $\binom{\mathbf{C}}{\mathbf{A}}$ there exists $f \in \binom{\mathbf{C}}{\mathbf{B}}$ such that $\binom{f(\mathbf{B})}{\mathbf{A}}$ has at most t colours.

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Sierpiński: not true for $|O| = 2$.

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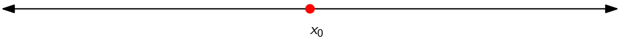

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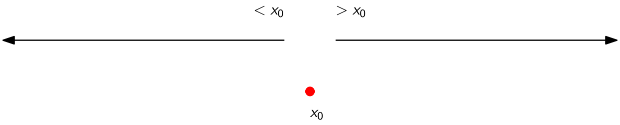

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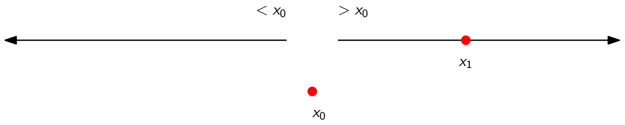

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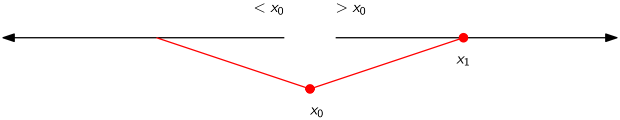

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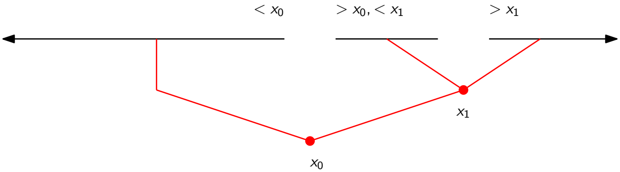

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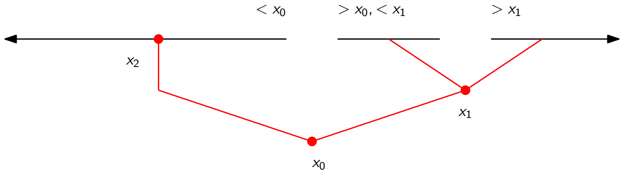

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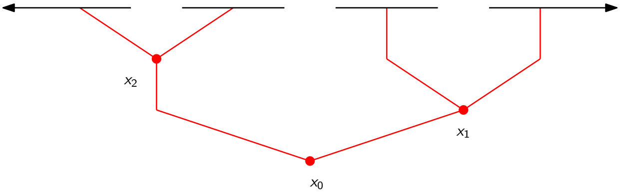

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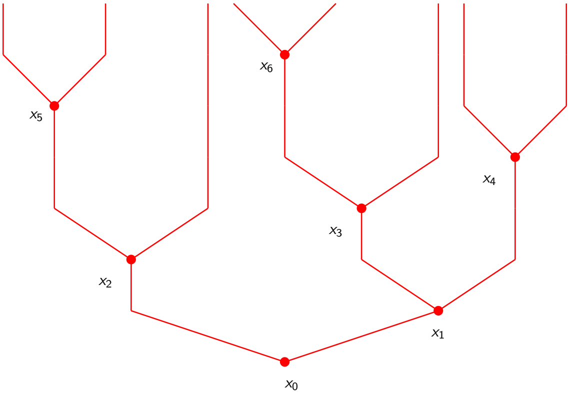

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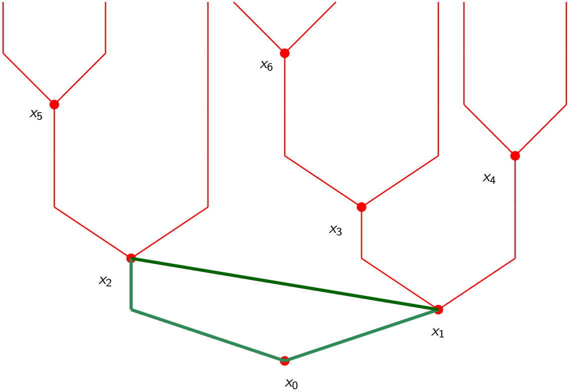

Colour of k-tuple = shape of meet closure in the tree

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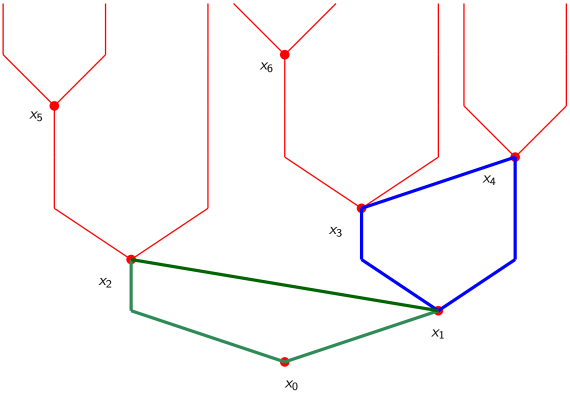

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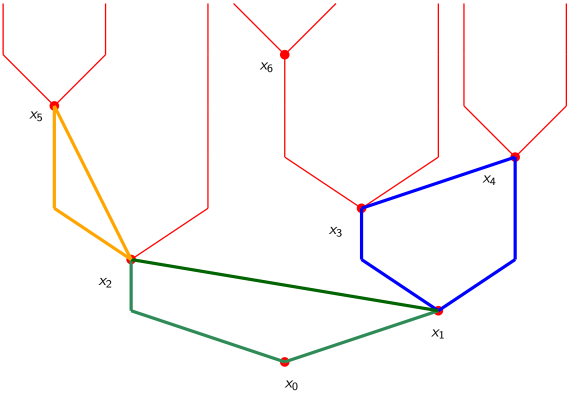

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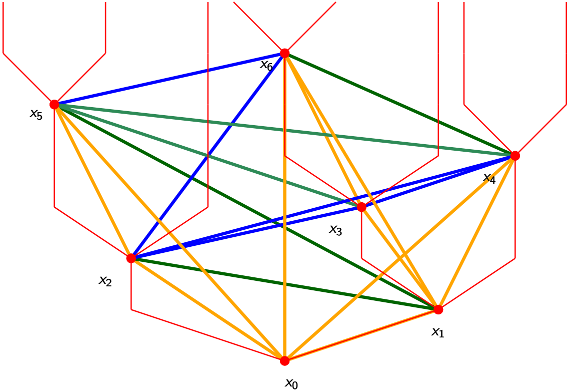

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In late 1960's Laver developed method of finding copies of $\mathbb{Q}$ in $\mathbb{Q}$ with bounded number of colours using Milliken's tree theorem.

Theorem (Devlin, 1979)

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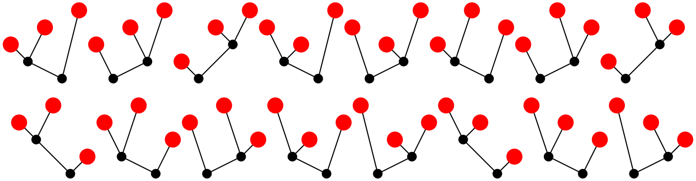


Upper bounds

Big picture: proof techniques

Ramsey's Theorem

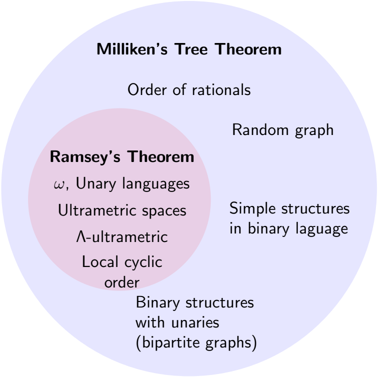
ω, Unary languages

Ultrametric spaces

Λ-ultrametric

Local cyclic order

Big picture: proof techniques



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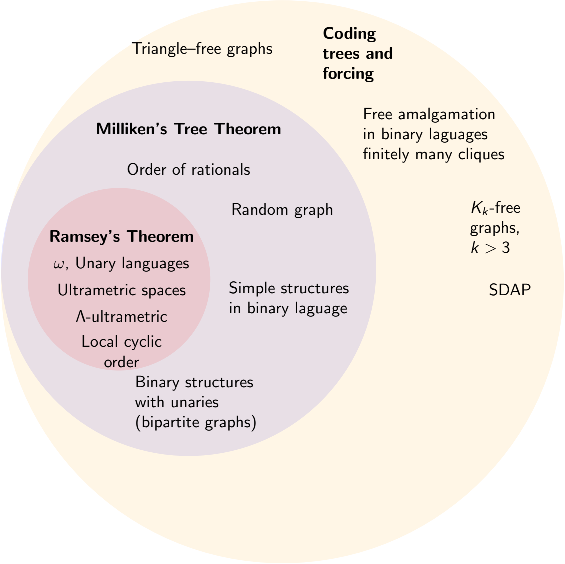

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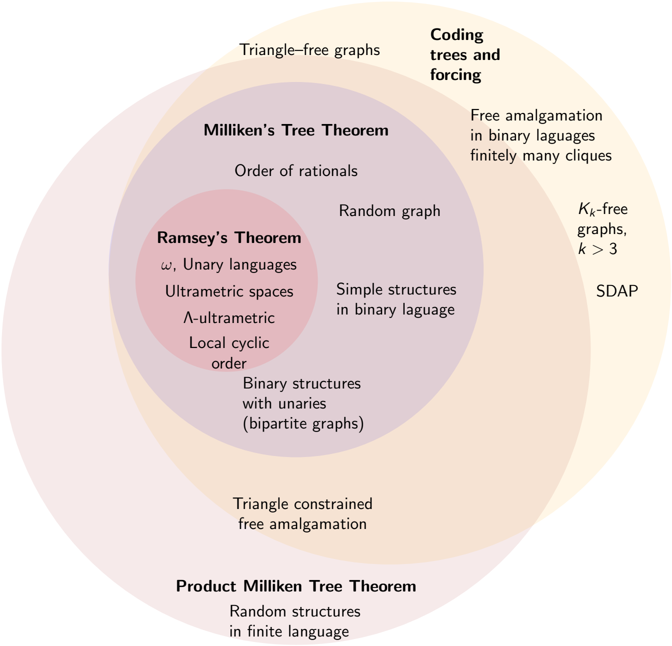

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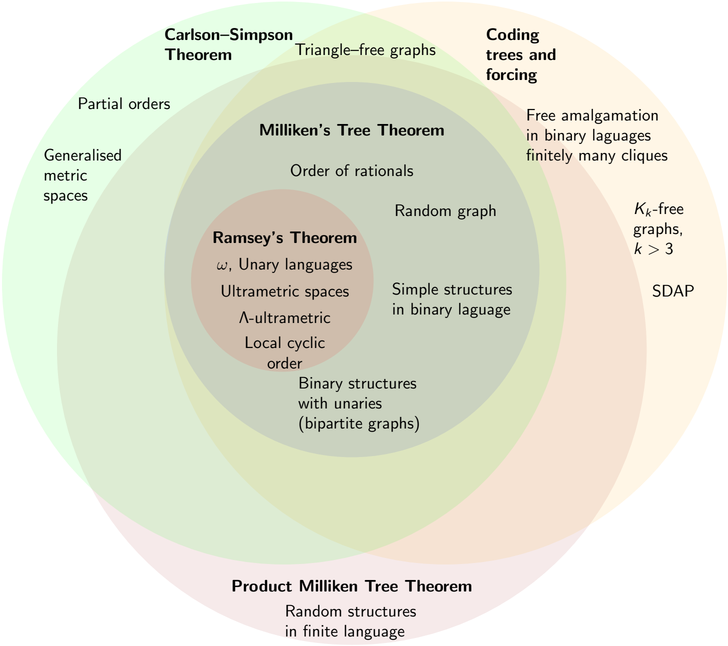
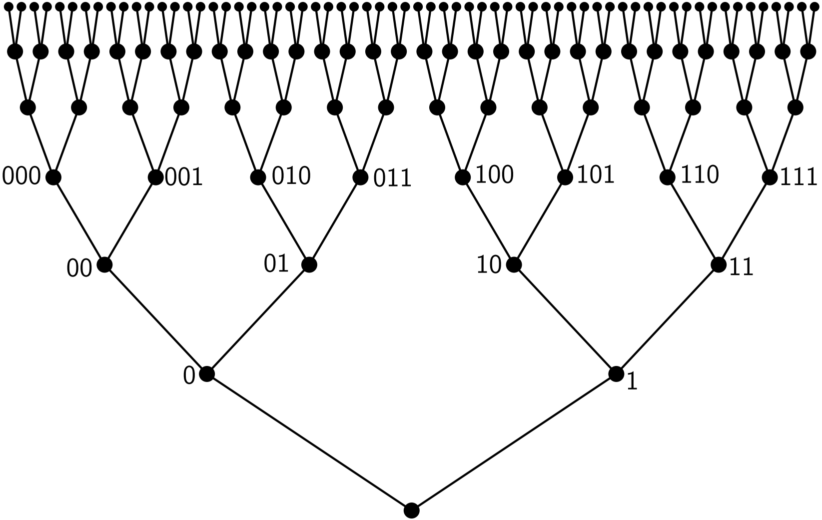
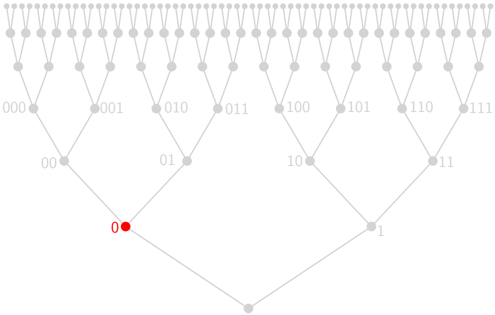


Tree



Strong subtree



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1. S has a root node.

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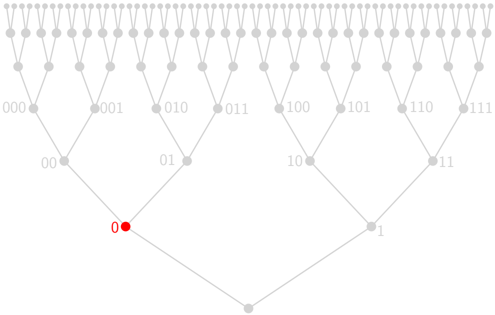

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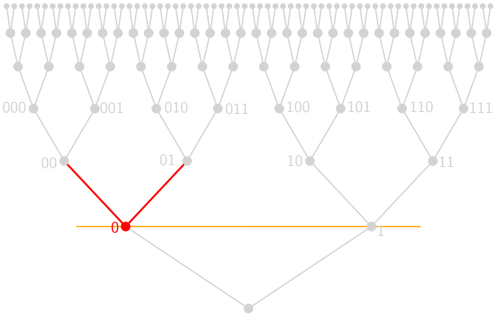

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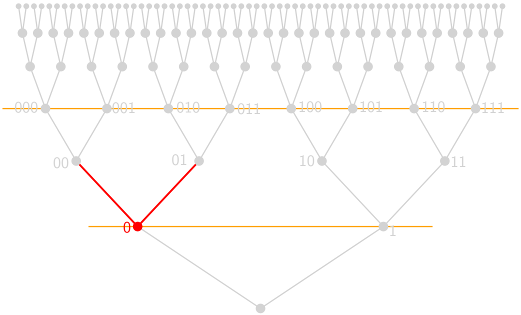

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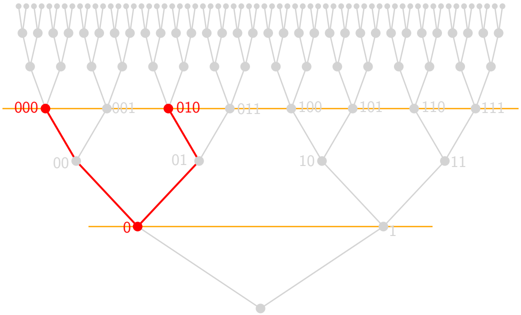

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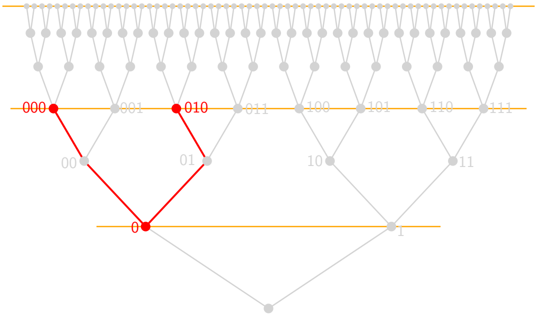

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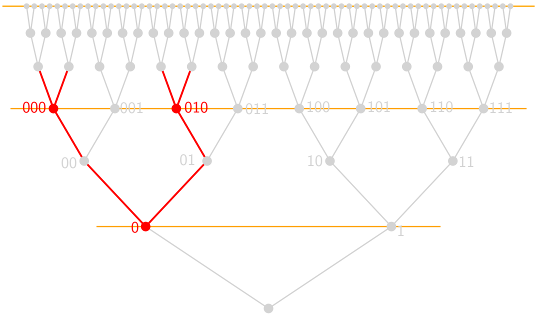

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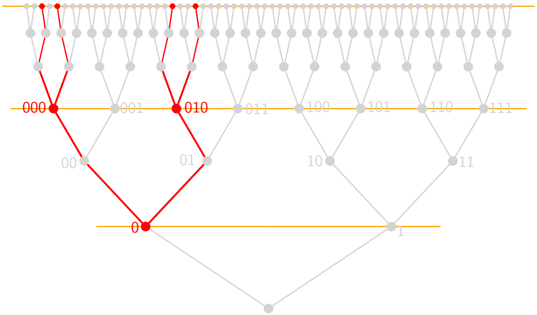

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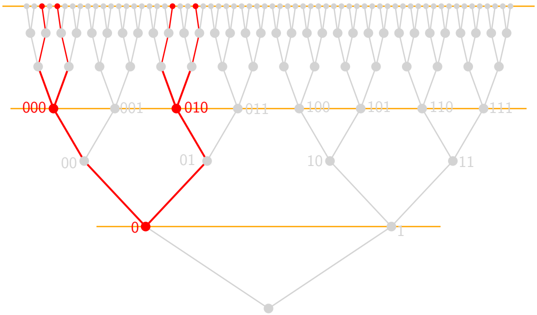

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3. Every level of S is a subset of some level of T.
4. S either has no leaves or all are at the same level.

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Theorem (Milliken tree theorem 1979)

For every infinitely branching tree $\mathbf{T}$ *with no leaves,* $k \geq 1$*, and every finite colouring of strong subtrees of depth* k *there exists infinite strong subtree* $\mathbf{S} \subseteq \mathbf{T}$ *which is monochromatic.*

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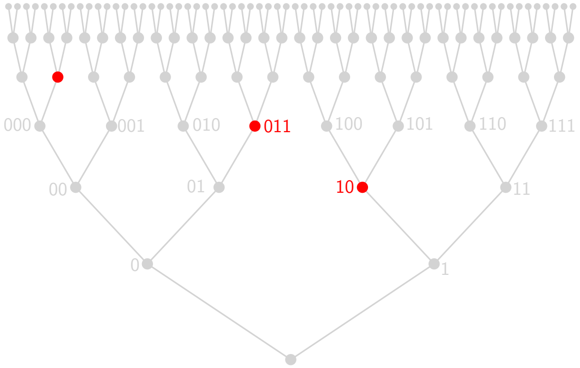
4. By Milliken tree theorem there is infinite monochromatic subtree of **B**. This subtree ordered by $\sqsubseteq$ contains a copy of **Q**. Colour of a copy of **O** depends only on a shape within envelope, the embedding type.

□

Envelopes of subsets



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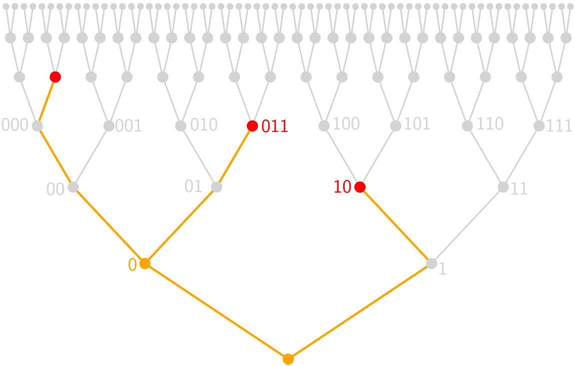

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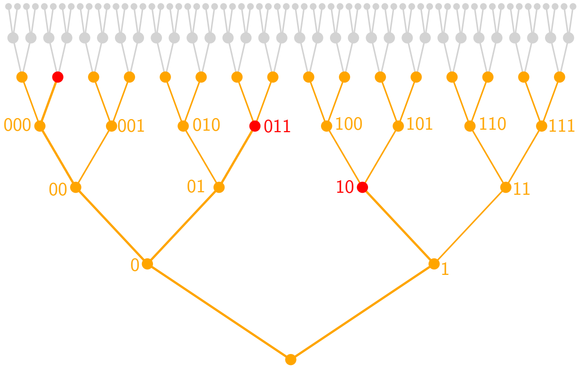

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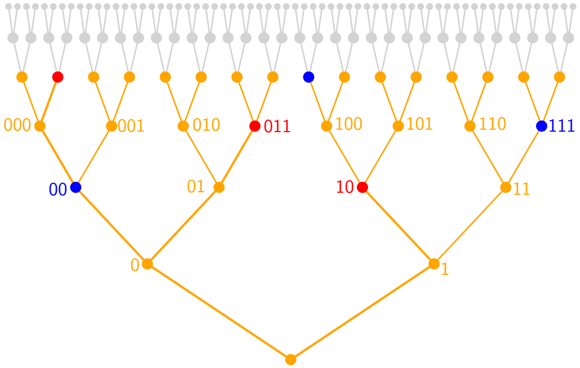

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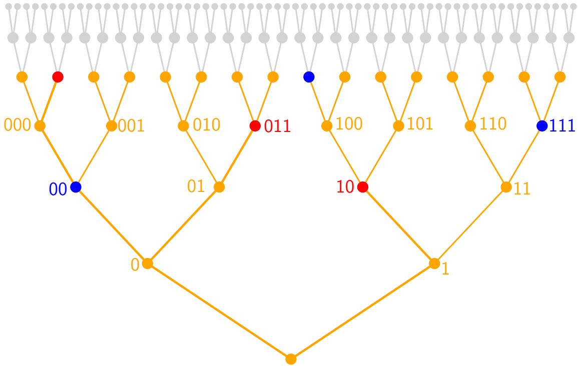

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Main observation

Number of sets yielding a given envelope is bounded. Based on finite coloring of subsets of a given size k it is possible to produce finite coloring of envelopes and apply Milliken tree theorem to find a copy with bounded number of colors

Big Ramsey degrees of $(P, \leq)$

Let $\mathcal{P}$ be the class of all finite partial orders.

Theorem (J. H. 2020+)

Every (countable) universal partial order $(P, \leq)$ has finite big Ramsey degrees:

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Universality: every countable partial order has embedding to $(P, \leq)$.

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- Construction generalizes to large family of classes that are described by forbidden induced cycles in particular to metric spaces with bounded number of distances (joint work with M. Balko, D. Chodounský, N. Dobrinen, M. Konečný, J. Nešetřil, L. Vena, A. Zucker)
- Big Ramsey degrees of posets and triangle-free graphs was fully characterised (joint work with M. Balko, D. Chodounský, N. Dobrinen, M. Konečný, L. Vena, A. Zucker
- Generalizations to bigger forbidden cliques and higher arities is work in progress

Tree of types of a universal triangle-free graph

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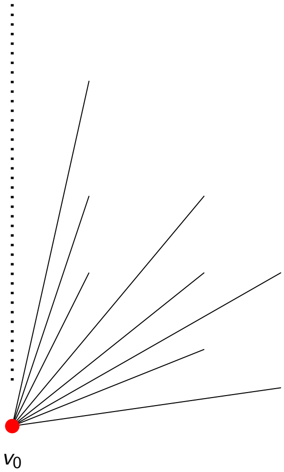

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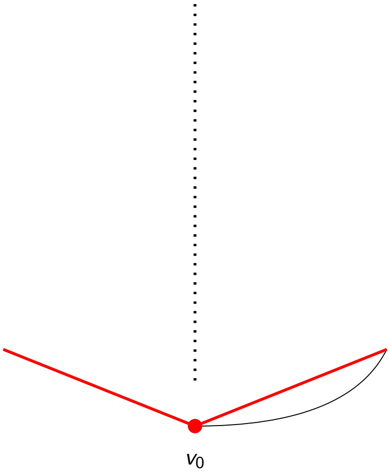

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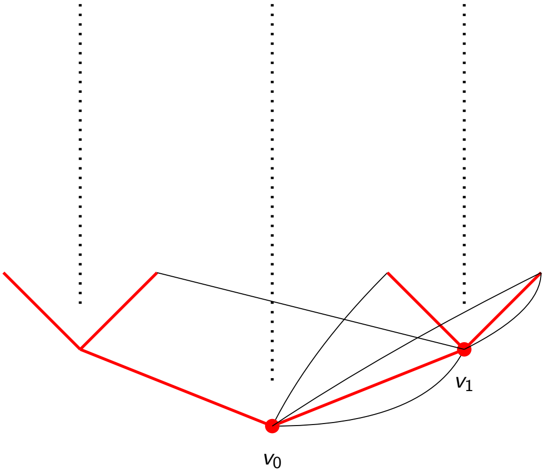

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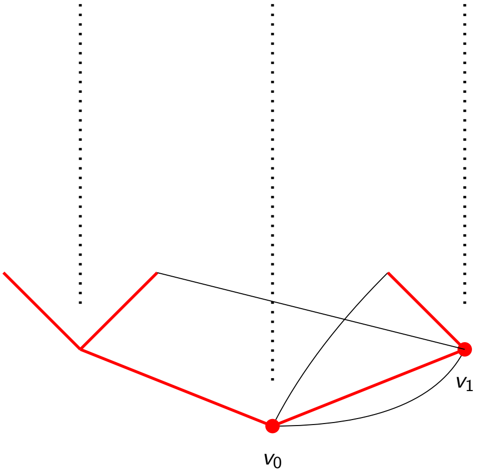

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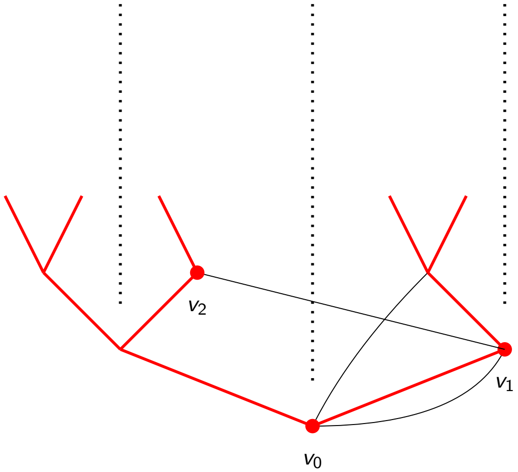

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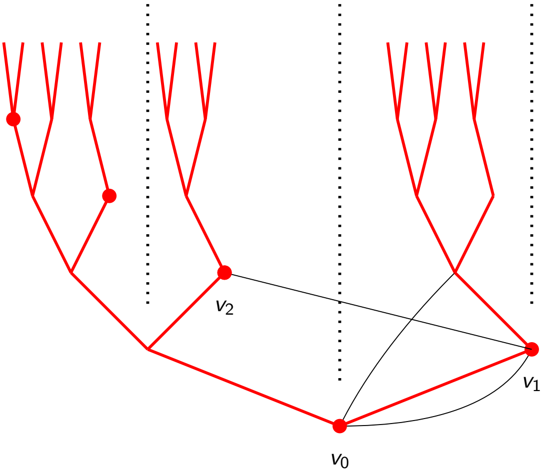

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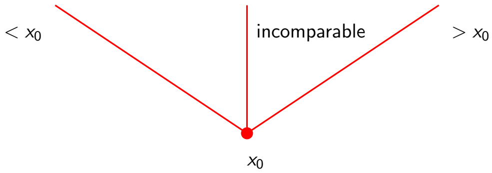

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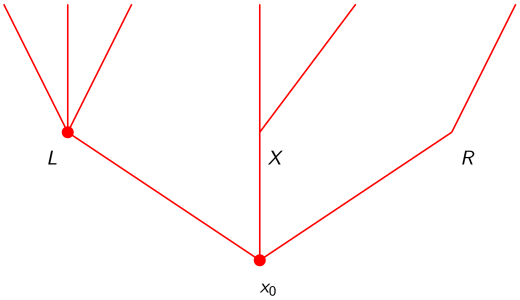

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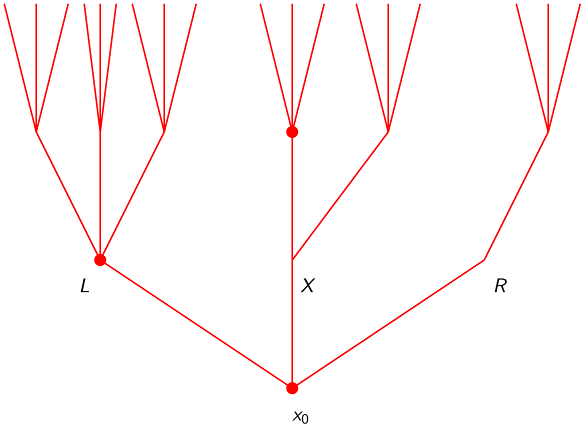

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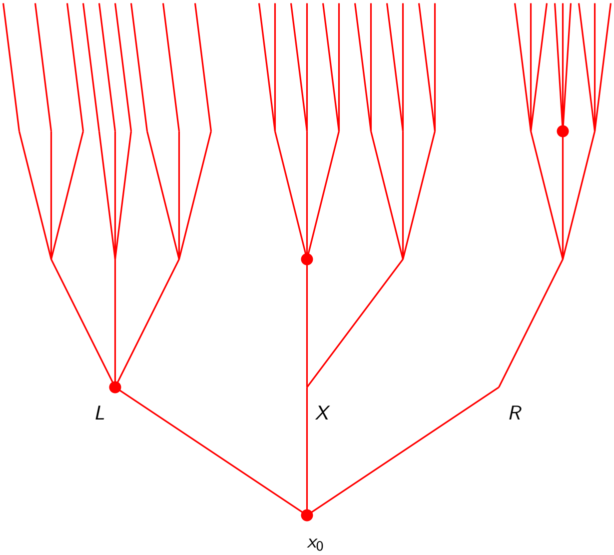

Parameter words

Definition (Parameter word)

Given a finite alphabet Σ and $k \in \omega + 1$, a *k-parameter word* is a (possibly infinite) word W in alphabet $\Sigma \cup \{\lambda_i : 0 \leq i < k\}$ such that $\forall i \in k$ word W contains λ_i and for every $j \in k - 1$, the first occurrence of λ_{j+1} appears after the first occurrence of λ_j.

Example (2-parameter word)

$\Sigma = \{\mathrm{L}, \mathrm{X}, \mathrm{R}\}$.

$$\mathrm{LRL}\lambda_0\lambda_0\mathrm{X}\lambda_1\lambda_0\mathrm{R}$$

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For set S of parameter words and a parameter word W:

$$W(S) = \{W(U) : U \in S\}.$$

Ramsey theorem for parameter words

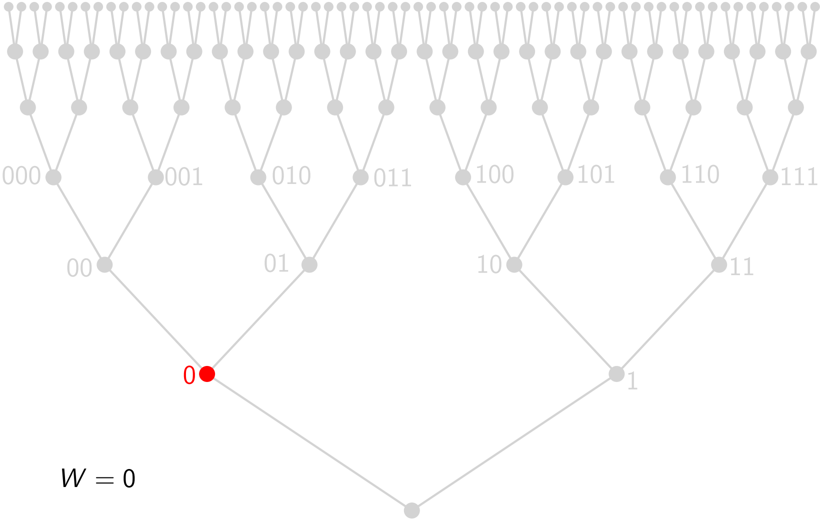
The following infinitary version of Graham–Rothschild Theorem is a direct consequence of the Carlson–Simpson theorem. It was also independently proved by Voight in 1983 (apparently unpublished):

Theorem (Ramsey theorem for parameter words)

Let Σ be a finite alphabet and $k \geq 0$ a finite integer. If the set of all finite k-parameter words in alphabet Σ is coloured by finitely many colours, then there exists a monochromatic infinite-parameter word W.

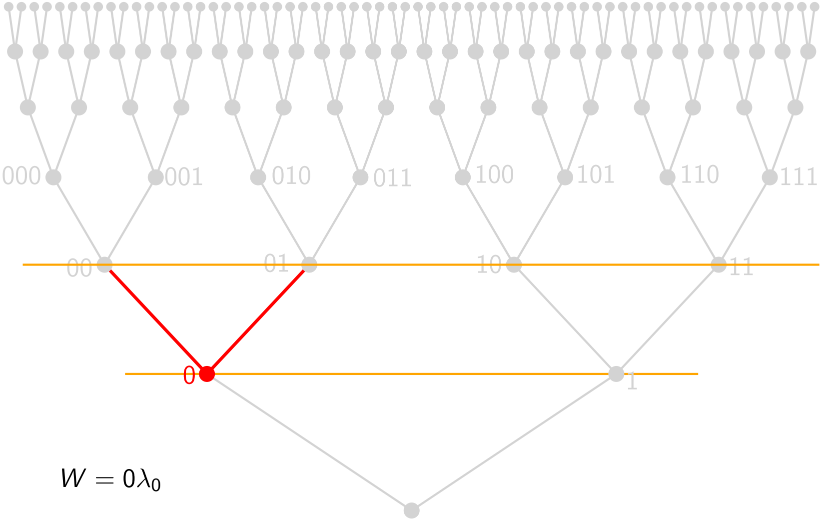
By W being monochromatic we mean that for every pair of k-parameter words U, V the colour of $W(U)$ is the same as colour of $W(V)$.

Parameter words as subtrees



$W = 0$

Parameter words as subtrees



$W = 0\lambda_0$

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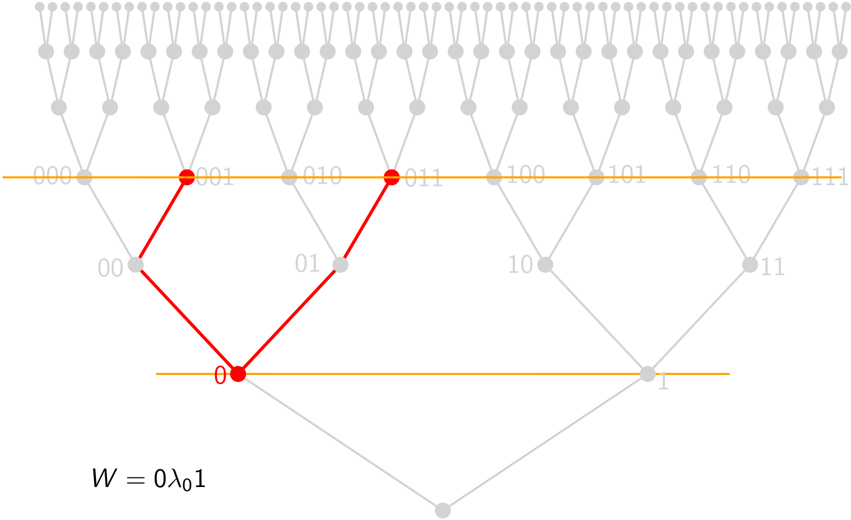

$W = 0\lambda_0 1$

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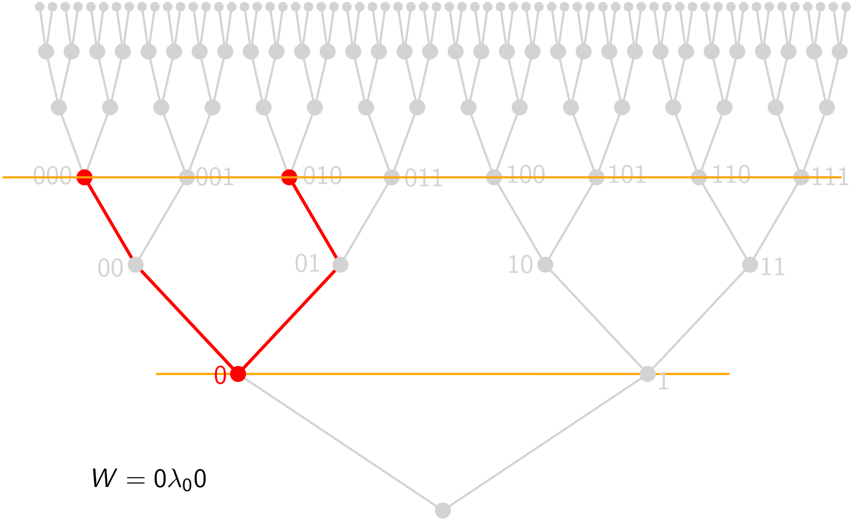

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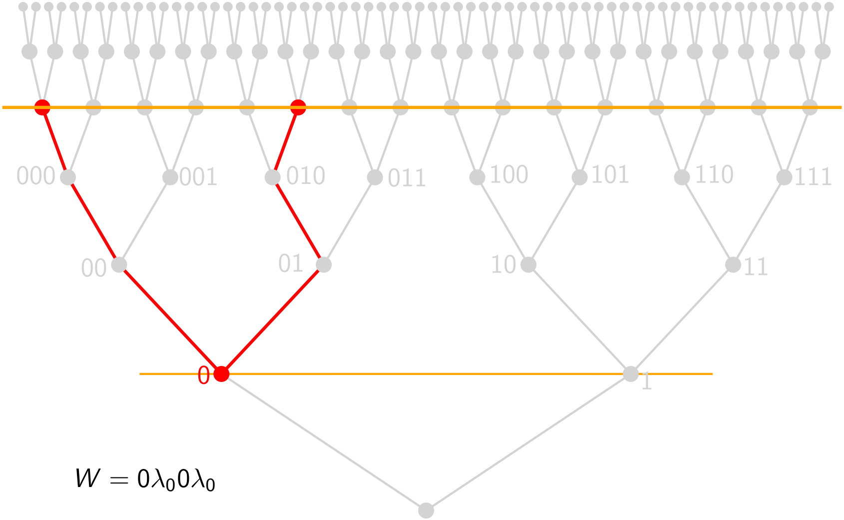

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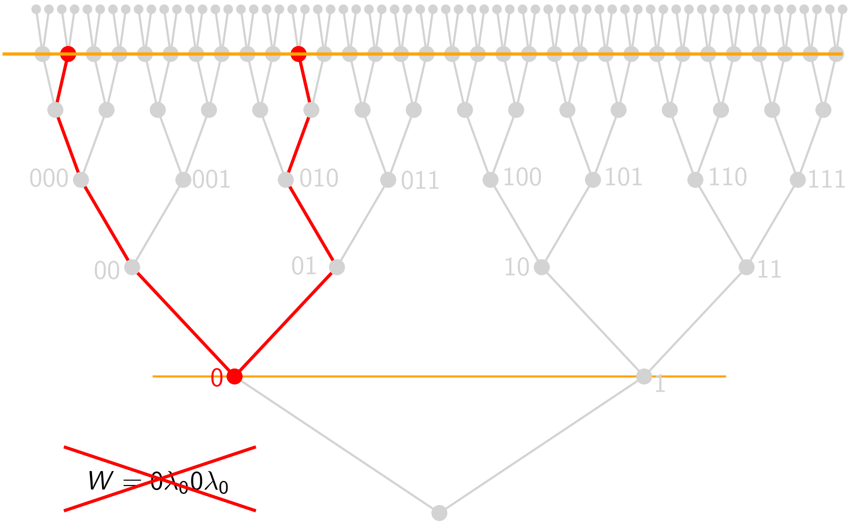

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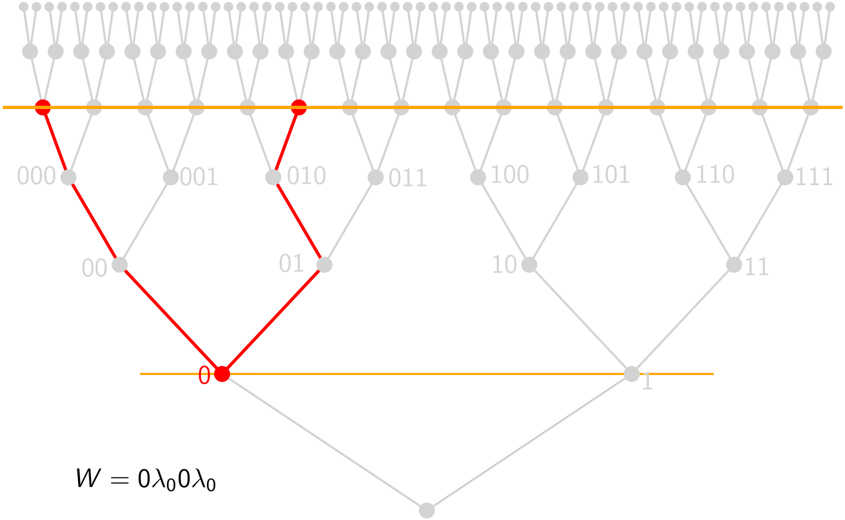

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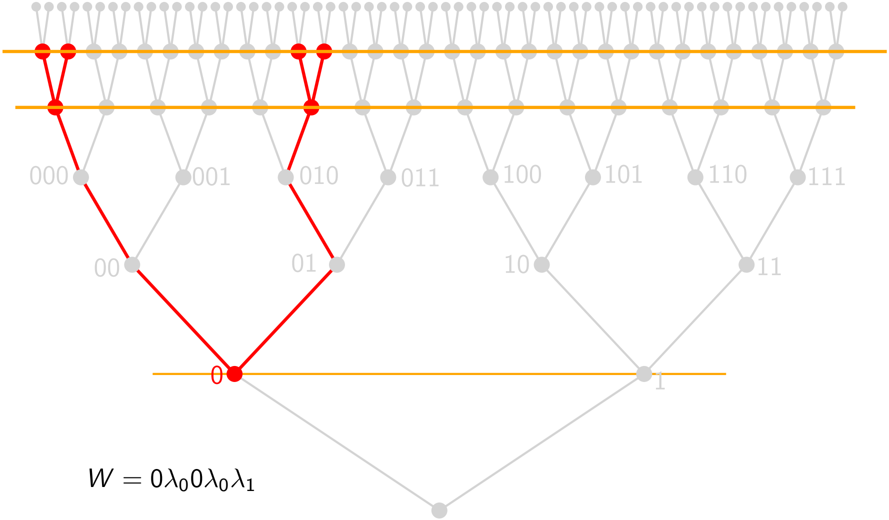

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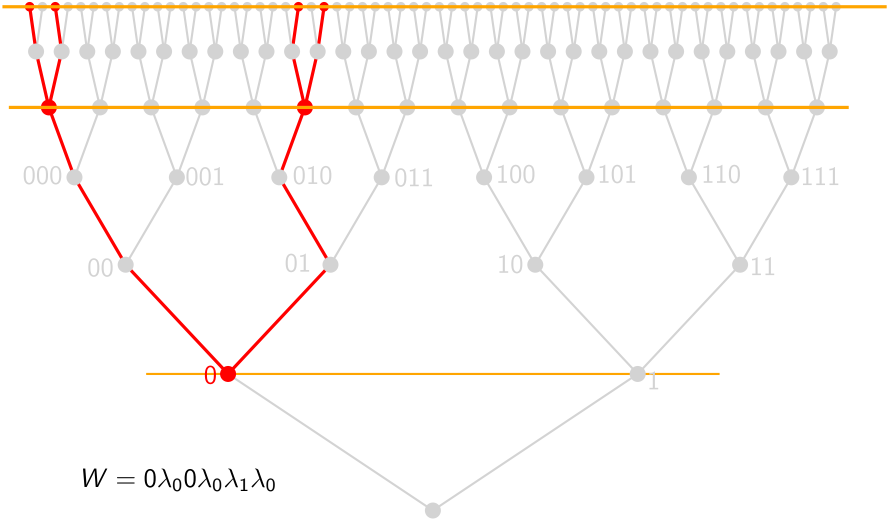

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Definition

Given a finite alphabet Σ, a finite integer $k \geq 0$ and a finite set k-parameter words, an envelope of S is every n-parameter word W (for some $n \geq k$) such that

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Proof.

$$\begin{array}{lccccccc} U & = & 0 & 1 & 1 & 0 & 1 & \\ V & = & 0 & 0 & 1 & 1 & 0 & 1 \end{array}$$

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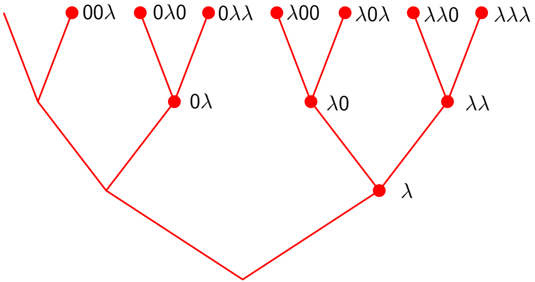
□

Triangle-free graph on 1-parameter words



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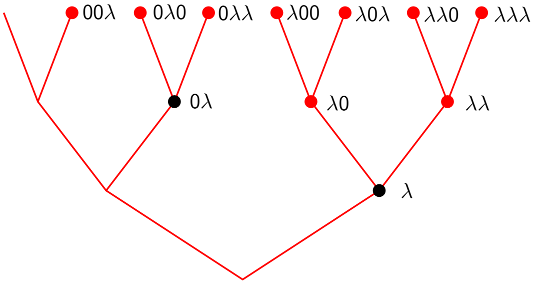

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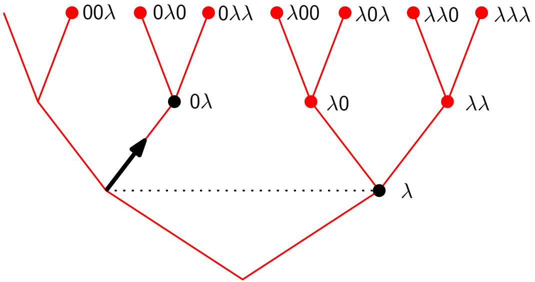

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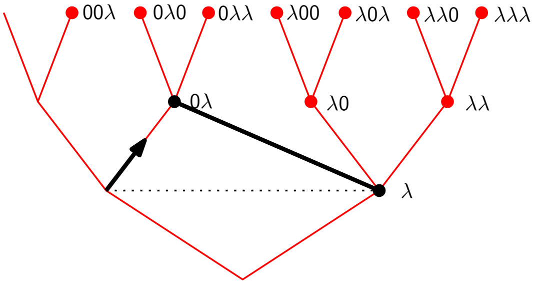

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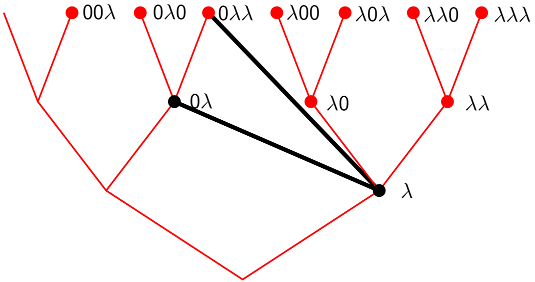

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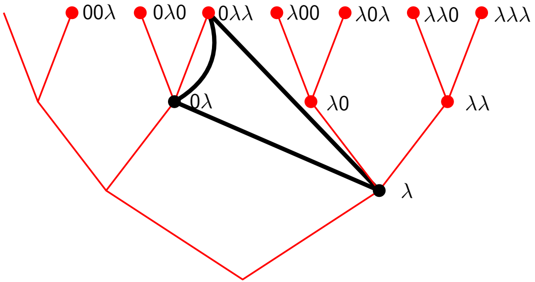

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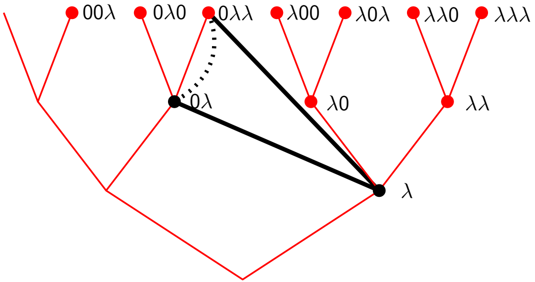

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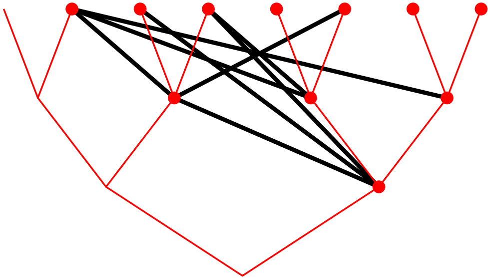

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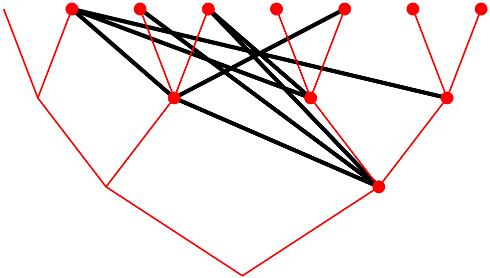

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Key observation 1: G is universal triangle-free graph.

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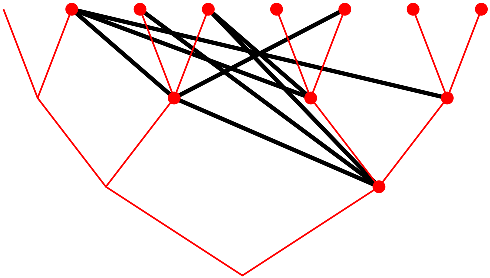

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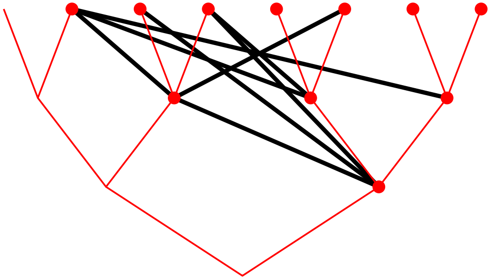
- For 1-parameter words u, v satisfying $|u| < |v|$ we put $u \sim v$ (u adjacent to v) iff
  1. $V_{|U|} = \lambda$ and
  2. for no $i \in |u|$ it holds that $U_i = V_i = \lambda$.

Key observation 1: G is universal triangle-free graph.
Given any triangle-free graph H with vertex set ω assign every $i \in \omega$ word w of length i putting $\forall_{j<i}\ w_j = \lambda$ iff $\{i, j\}$ is an edge of H.

Triangle-free graph on 1-parameter words



Put $\Sigma = \{0\}$.

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Key observation 1: G is universal triangle-free graph.

Key observation 2: For every pair of 1-parmeter words U and V and every ω-parameter W

$$U \sim V \iff W(U) \sim W(V).$$

Observation

G is a universal triangle-free graph.

Observation

For every infinite-parameter word W it holds that $u \sim v \iff W(u) \sim W(v)$.
(Substitution is also graph embedding on $G \to G$.)

Theorem (Ramsey theorem for parameter words)

Let Σ be a finite alphabet and $k \geq 0$ a finite integer. If the set of all finite k-parameter words in alphabet Σ is coloured by finitely many colours, then there exists a monochromatic infinite-parameter word W.

Proposition (Envelopes are bounded)

There exists $T(|\Sigma|, s, k)$ such that for every set S of size s of k-parameter words in alphabet Σ there exists an envelope of S with at most $T(|\Sigma|, s, k)$ parameters.

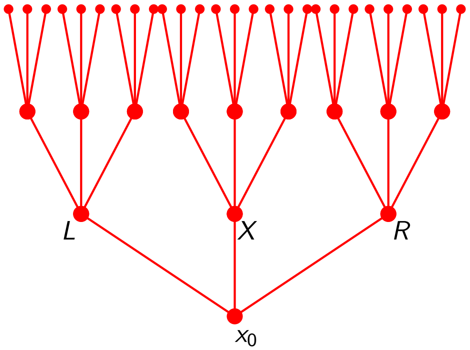
Theorem (Dobrinen 2020)

The big Ramsey degrees of universal triangle-free graph are finite.

Proof.

Fix graph A and a finite coloring of $\binom{G}{A}$. Because envelopes of copies of A are bounded, apply the theorem above for every embedding type and obtain a copy of G with bounded number of colors. $\square$

Partial order on infinite ternary tree



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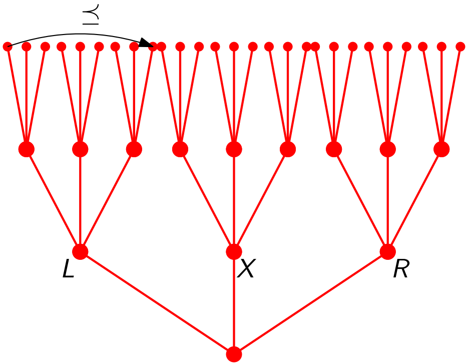

Put $\Sigma = \{\mathrm{L}, \mathrm{X}, \mathrm{R}\}$ and order $\mathrm{L} <_{\mathrm{lex}} \mathrm{X} <_{\mathrm{lex}} \mathrm{R}$.

Definition (Partial order $(\Sigma^*, \preceq)$)

For $w, w' \in \Sigma^*$ we put $w \prec w'$ if and only if there exists $0 \leq i < \min(|w|, |w'|)$ such that

1. $(w_i, w'_i) = (\mathrm{L}, \mathrm{R})$ and
2. for every $0 \leq j < i$ it holds that $w_j \leq_{\mathrm{lex}} w'_j$.

Key observations: $\preceq$ is universal partial order and is stable for substitution.

Lower bounds

Devlin-type

Given n, the big Ramsey degree of linear order of size n is to the number of Devlin-types.

Notation:

- Σ^* is the set of all finite words in alphabet Σ.
- Given $S \subseteq \Sigma^*$ by $\overline{S}$ we denote the set of all initial segments of words in S.
- By $\overline{S}_i$ we denote the set of all initial segments of S of length i.
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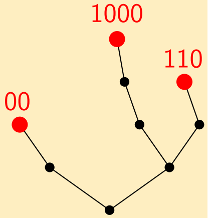

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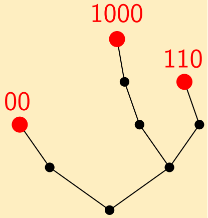

$$t_1 = 1, t_n = \sum_{\ell=1}^{n-1} \binom{2n-2}{2\ell-1} t_\ell \cdot t_{n-\ell}.$$

Level-structures

Definition

Given level ℓ of the tree of types, we can consider its level-structure:

1. Vertices are nodes (types) of level ℓ.
2. We write $a \preceq b$ if it is true that $a' \leq b'$ for every successor a' of a and b' of b.
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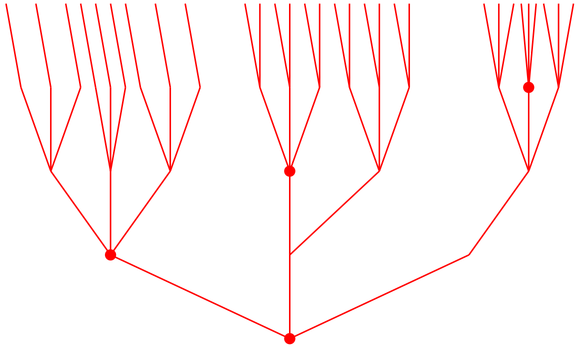

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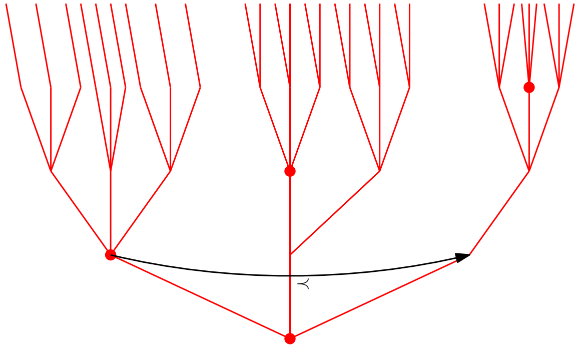

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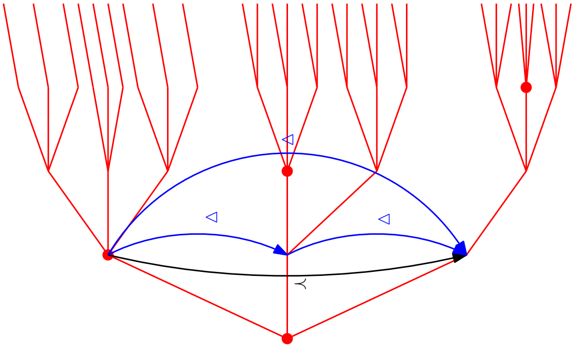

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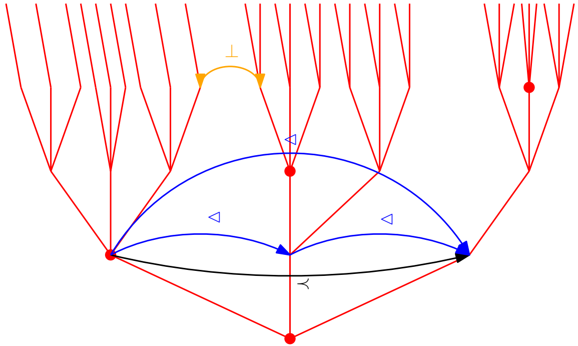

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Fun fact

It turns out that both $\preceq$ and $\trianglelefteq$ are partial orders and whenever $a \preceq b$ also $a \trianglelefteq b$. One can think of the level structure $(A, \preceq, \trianglelefteq)$ as of an finite approximation of the infinite partial order located above the given level.

Definition (Poset-type)

A set $S \subseteq \{\mathrm{L}, \mathrm{X}, \mathrm{R}\}^*$ is called a poset-type if precisely one of the following four conditions is satisfied for every level ℓ with $0 \leq \ell < \max_{w \in S} |w|$:

1. **Leaf:** There is $w \in \overline{S}_\ell$ related to every $u \in \overline{S}_\ell \setminus \{w\}$ and $\overline{S}_{\ell+1} = (\overline{S}_\ell \setminus \{w\})^\frown \mathrm{X}$.
2. **Branching:** There is $w \in \overline{S}_\ell$ such that
$$\overline{S}_{\ell+1} = \{z \in \overline{S}_\ell : z <_{\text{lex}} w\}^\frown \mathrm{X} \cup \{w^\frown \mathrm{X}, w^\frown \mathrm{R}\} \cup \{z \in \overline{S}_\ell : w <_{\text{lex}} z\}^\frown \mathrm{R}.$$
3. **New $\perp$:** There are unrelated words $v <_{\text{lex}} w \in \overline{S}_\ell$ such that
$$\begin{aligned}\overline{S}_{\ell+1} = &\{z \in \overline{S}_\ell : z <_{\text{lex}} v\}^\frown \mathrm{X} \cup \{v^\frown \mathrm{R}\} \cup \{z \in \overline{S}_\ell : v <_{\text{lex}} z <_{\text{lex}} w \text{ and } z \perp v\}^\frown \mathrm{X} \\ &\cup \{z \in \overline{S}_\ell : v <_{\text{lex}} z <_{\text{lex}} w \text{ and } z \not\perp v\}^\frown \mathrm{R} \cup \{w^\frown \mathrm{X}\} \cup \{z \in \overline{S}_\ell : w <_{\text{lex}} z\}^\frown \mathrm{R}.\end{aligned}$$
   Moreover for every $u \in \overline{S}_\ell$, $v <_{\text{lex}} u <_{\text{lex}} w$ implies that at least one of $u \perp v$ or $u \perp w$ holds.
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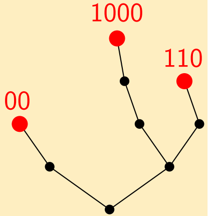

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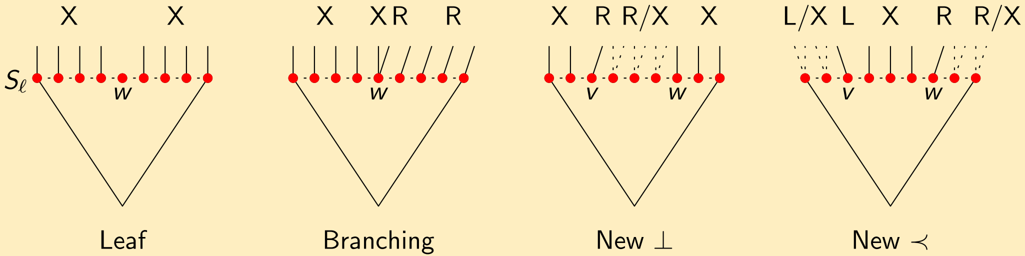

Main result

Given a finite partial order $(A, \leq)$, we let $T(A, \leq)$ be the set of all poset-types S such that $(S, \preceq)$ is isomorphic to $(A, \leq)$.

Theorem (M. Balko, D. Chodounský, N. Dobrinen, J. H., M. Konečný, L. Vena, A. Zucker)

For every finite partial order $(O, \leq)$, the big Ramsey degree of $(O, \leq)$ in the universal partial order $(P, \leq)$ equals $|T(O, \leq)| \cdot |\mathrm{Aut}(O, \leq)|$.

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Example

Denote by $\mathbf{A}_n$ the anti-chain with with n vertices and by $\mathbf{C}_n$ the chain with n vertices.

$$T(\mathbf{A}_1) = T(\mathbf{C}_1) = \{\emptyset\}$$

$$T(\mathbf{A}_2) = \{\{\mathrm{XR}, \mathrm{RXX}\}, \{\mathrm{XRX}, \mathrm{RX}\}\}$$
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Overall there are:

- 1 poset-types a vertex,
- 4 poset-types of posets of size 2,
- 464 poset-types of posets of size 3,
- 1874880 poset-types of posets of size 4.

Approximating posets

Definition (Approximate poset)

An approximate poset is a structure $(A, \preceq, \trianglelefteq, P)$ where

1. both $\preceq$ and $\trianglelefteq$ are partial order.
2. P is an unary predicate denoting "finished vertices".
3. $\preceq \subseteq \trianglelefteq$ (whenever $a \preceq b$ also $a \trianglelefteq b$).
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Approximating posets

$\preceq$ is black, $\trianglelefteq$ is blue
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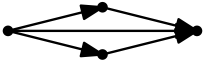
Goal



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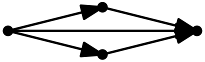

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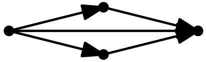

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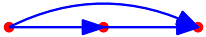

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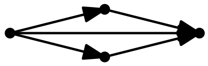

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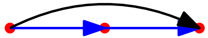

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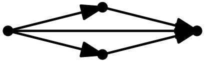

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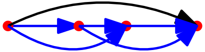

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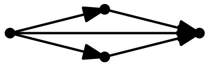

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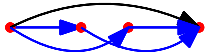

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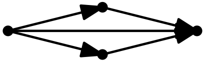

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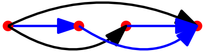

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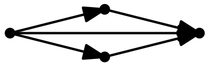

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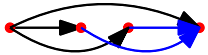

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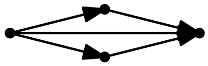

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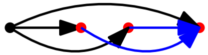

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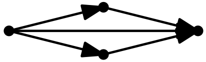

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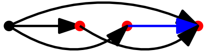

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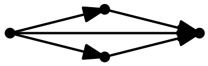

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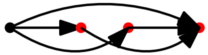

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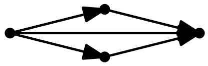

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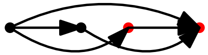

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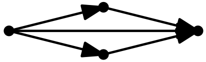

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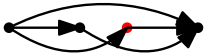

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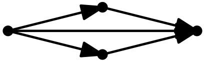

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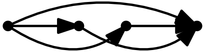

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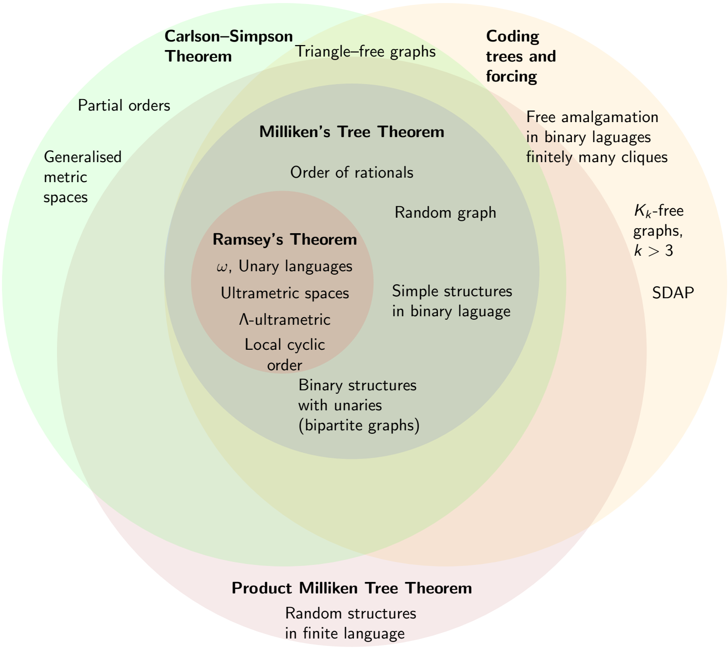
Work in progress

1. Generalizations to metric spaces and other strong amalgamation classes
2. Abstract theorem for trees with successor operations
3. New proof of upper bounds of big Ramsey degrees of free amalgamation classes in binary language
4. Generalizations to higher arity

Thank you for the attention



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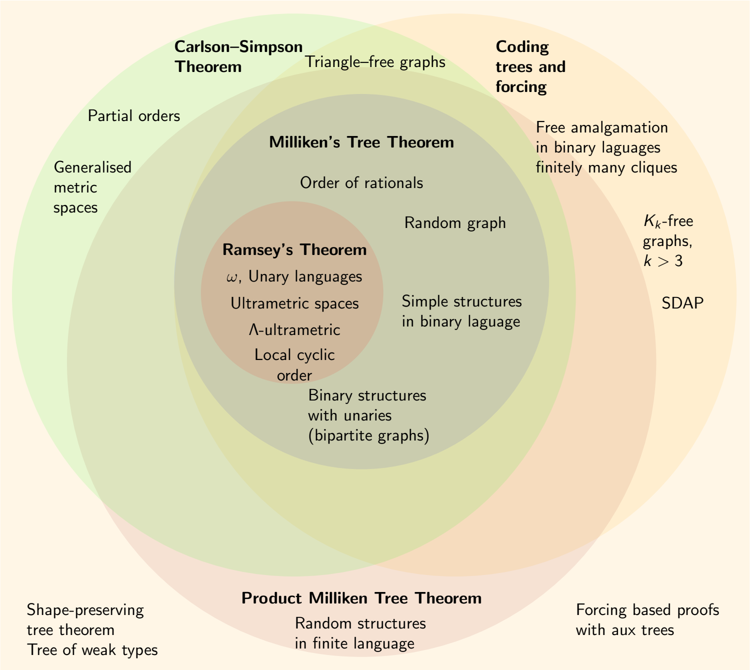

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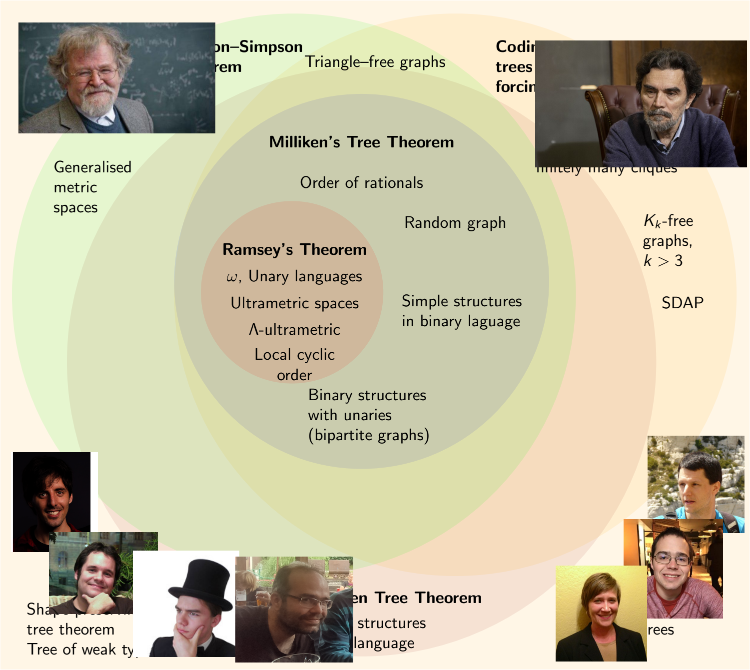

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