Ramsey Classes by Partite Construction I

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Ramsey classes

We consider relational structures in language $L$ without function symbols.

**Definition**

A class $C$ (of finite relational structures) is **Ramsey** iff

$$\forall_{A,B \in C} \exists_{C \in C} : C \rightarrow (B)_2^A.$$
We consider relational structures in language $L$ without function symbols.

**Definition**

A class $C$ (of finite relational structures) is **Ramsey** iff

$$\forall_{A, B \in C} \exists C \in C : C \rightarrow (B)^A_2.$$ 

$(B)^A_2$ is the set of all substructures of $B$ isomorphic to $A$.

$C \rightarrow (B)^A_2$: For every 2-coloring of $(C)^A_2$ there exists $\tilde{B} \in (B)^C_B$ such that $(\tilde{B})^A_A$ is monochromatic.
We consider relational structures in language $L$ without function symbols.

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A class $\mathcal{C}$ (of finite relational structures) is **Ramsey** iff

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We consider relational structures in language $L$ without function symbols.

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Examples of Ramsey classes

Example
The class of all finite linear orders is Ramsey.
Examples of Ramsey classes

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The class of all finite linear orders is Ramsey.

...
Examples of Ramsey classes

**Example**
The class of all finite linear orders is Ramsey.

... 

**Example (Non-example)**
The class of all directed graphs is not Ramsey.
Examples of Ramsey classes

Example

The class of all finite linear orders is Ramsey.

... 

Example (Non-example)

The class of all directed graphs is not Ramsey.

\begin{center}
\begin{tikzpicture}
  \node (A) at (0,0) {A};
  \node (B) at (1,1.732) {B};
  \draw [blue, very thick, ->] (A) -- (B);
  \draw [blue, very thick, ->] (B) -- (A);
\end{tikzpicture}
\end{center}

Given \textbf{C} consider arbitrary linear order. Color edges red if they go forward in the linear order and blue otherwise.
Definition (Amalgamation property of class $\mathcal{K}$)

Ramsey classes are amalgamation classes.
Ramsey classes are amalgamation classes

Definition (Amalgamation property of class $\mathcal{K}$)

Nešetřil, 1989: Under mild assumptions Ramsey classes have amalgamation property.
Classification programme

Ramsey classes $\Rightarrow$ amalgamation classes

$\uparrow$

lifts of homogeneous $\iff$ homogeneous structures

Many amalgamation classes are given by the classification programme of homogeneous structures.

Can we always find a Ramsey lift?

Theorem (Nešetříl, 1989)
All homogeneous graphs have Ramsey lift.

Theorem (Jasiński, Laflamme, Nguyen Van Thé, Woodrow, 2014)
All homogeneous digraphs have Ramsey lift.
### Classification programme

<table>
<thead>
<tr>
<th>Ramsey classes</th>
<th>$\implies$</th>
<th>amalgamation classes</th>
</tr>
</thead>
<tbody>
<tr>
<td>lifts of homogeneous</td>
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$\uparrow$
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Map of Ramsey Classes

free

restricted

linear orders

cyclic orders

unions of complete graphs

interval graphs

permutations

J. Hubička

Ramsey Classes by Partite Construction I
A structure $A$ is called complete (or irreducible) if every pair of distinct vertices belong to a relation of $A$.

$\text{Forb}_E(\mathcal{E})$ is a class of all finite structures $A$ such that there is no embedding from $E \in \mathcal{E}$ to $A$.

**Theorem (Nešetřil-Rödl Theorem, 1977)**

- Let $L$ be a finite relational language.
- Let $\mathcal{E}$ be a set of complete ordered $L$-structures.
- The then class $\text{Forb}_E(\mathcal{E})$ is a Ramsey class.
A structure $A$ is called complete (or irreducible) if every pair of distinct vertices belong to a relation of $A$.

$\text{Forb}_E(\mathcal{E})$ is a class of all finite structures $A$ such that there is no embedding from $E \in \mathcal{E}$ to $A$.

**Theorem (Nešetřil-Rödl Theorem, 1977)**

- Let $L$ be a finite relational language.
- Let $\mathcal{E}$ be a set of complete ordered $L$-structures.
- The then class $\text{Forb}_E(\mathcal{E})$ is a Ramsey class.

Explicitly: For every $A, B \in \text{Forb}_E(\mathcal{E})$ there is $C \in \text{Forb}_E(\mathcal{E})$ such that $C \twoheadrightarrow (B)^A_2$. 
Examples of Ramsey lifts

Example

Graphs with order are Ramsey.

\begin{figure}
\centering
\includegraphics[width=\textwidth]{example_graph}
\caption{Example graph with order.}
\end{figure}
Examples of Ramsey lifts

Example
Graphs with order are Ramsey.

Example
Acyclic graphs with linear extension are Ramsey.
Examples of Ramsey lifts

**Example**
Graphs with order are Ramsey.

![Graph](image1)

**Example**
Acyclic graphs with linear extension are Ramsey.

![Graph](image2)

**Example**
Bipartite graphs with unary relation identifying bipartition and with convex linear order are Ramsey.

![Graph](image3)
Map of Ramsey Classes

interval graphs

permutations
cyclic orders
unions of complete graphs
linear orders

restricted

free
Map of Ramsey Classes

interval graphs
permutations
cyclic orders
unions of complete graphs
linear orders

restricted

metric spaces
acyclic graphs
partial orders
$K_n$-free graphs
graphs

free
Map of Ramsey Classes

- interval graphs
- permutations
- cyclic orders
- unions of complete graphs
- linear orders

restricted

- metric spaces
- acyclic graphs
- partial orders
- $K_n$-free graphs

free

J. Hubička
Ramsey Classes by Partite Construction I
J. Hubička

Ramsey Classes by Partite Construction I
Definition (\(A\)-partite system)

Let \(A\) be an ordered relational structure on vertices \(\{1, 2, \ldots, a\}\).

An \(A\)-partite system is a tuple \((A, X_B, B)\) where \(B\) is structure and \(X_B = \{X^1_B, X^2_B, \ldots, X^a_B\}\) partitions vertex set of \(B\) into \(a\) classes (\(X^i_B\) are called parts of \(B\)) such that:

1. ordering satisfies \(X^1_B < X^2_B < \ldots < X^a_B\);
2. mapping (projection) \(\pi\) which maps every \(x \in X^i_B\) to \(i\) \((i = 1, 2, \ldots, a)\) is a homomorphism;
3. every tuple in every relation of \(B\) meets every class \(X^i_B\) in at most one element.
The Partite Construction

Construction outline:

Put $n$ such

$n \rightarrow (|B|)^{|A|}_2$.

(For every coloring of $|A|$ tuples in $\{1, 2, \ldots n\}$ there exists monochromatic subset of size $|B|$). Here $n = 6$. 
The Partite Construction

\[ A = \bullet \quad B = \bullet \quad C = ? \]

Construction outline:

- Put \( n \) such
  \[ n \rightarrow (|B|)^{|A|}_2. \]
  (For every coloring of \(|A|\) tuples in \( \{1, 2, \ldots n\} \) there exists monochromatic subset of size \(|B|\)). Here \( n = 6 \).

- Picture 0: \(|K|_n\)-partite system \( P_0 \) s.t. for every coloring of copies of \( A \) in \( P_0 \) where the color of a copy \( \tilde{A} \) depends only on a projection \( \pi(\tilde{A}) \) there exists a monochromatic copy of \( B \).
The Partite Construction

\[ A = \begin{array}{c} \bullet \\ \bullet \end{array} \quad B = \begin{array}{c} \bullet \\ \bullet \end{array} \quad C = ? \]

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- Enumerate by \( A_1, \ldots A_N \) all possible projections of copies of \( A \) in \( P_0 \).
The Partite Construction

\[ A = \bullet \quad B = \bullet \quad C = ? \]

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- Pictures 1\ldots n: \( K_n \)-partite systems \( P_1, \ldots P_N \) s.t. for every coloring of copies of \( A \) in \( P_i \) there exists a copy of \( P_{i-1} \) where all copies of \( A \) with projection \( A_i \) are monochromatic.
The Partite Construction: Picture 0

Picture 0: $K_n$-partite system $P_0$ s.t. for every coloring of copies of $A$ in $P_0$ where the color of a copy $\tilde{A}$ depends only on a projection $\pi(\tilde{A})$ there exists a monochromatic copy of $B$.

\[ A = \quad B = \]

\[ \text{Diagram of the construction.} \]
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\[ A = \quad B = \]
The Partite Construction: Picture 1

**Picture 1:** $K_n$-partite system $P_1$ s.t. for every coloring of copies of $A$ in $P_1$ there exists a copy of $P_0$ where all copies of $A$ with projection $A_1$ are monochromatic.
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\[ A = \begin{array}{c}
\cdot \\
\cdot \\
\cdot 
\end{array} \quad B = \begin{array}{c}
\cdot \\
\cdot \\
\cdot 
\end{array} \]
Picture 1: $K_n$-partite system $P_1$ s.t. for every coloring of copies of $A$ in $P_1$ there exists a copy of $P_0$ where all copies of $A$ with projection $A_1$ are monochromatic.
Picture 2: $K_n$-partite system $P_2$ s.t. for every coloring of copies of $A$ in $P_2$ there exists a copy of $P_1$ where all copies of $A$ with projection $A_2$ are monochromatic.
The Partite Construction: Picture 2

Picture 2: $K_n$-partite system $P_2$ s.t. for every coloring of copies of $A$ in $P_2$ there exists a copy of $P_1$ where all copies of $A$ with projection $A_2$ are monochromatic.
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$A = \quad B =$
The Partite Construction: Picture 2

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$A = \begin{array}{c}
\bullet \\
\bullet \\
\end{array}$

$B = \begin{array}{c}
\bullet \\
\bullet \\
\end{array}$
The Partite Construction: Summary

- Ramsey Theorem:
  \[ K_n \rightarrow (K_{|B|})_2^{K_{|A|}} \]
The Partite Construction: Summary

- Ramsey Theorem:
  \[ K_n \rightarrow (K_{|B|})_{2}^{K_{|A|}} \]
- Construct \( P_0 \)

Enumerate by \( A_1, \ldots, A_N \) all possible projections of copies of \( A \) in \( P_0 \)

Construct \( P_1, \ldots, P_N \)

\( B_i \): partite system induced on \( P_i \) by all copies of all with projection to \( A_i \)

Partite lemma:

\[ C_i \rightarrow (B_i)_{A_i}^{2} \]

\( P_i \) is built repeated free amalgamation of \( P_i \) over all copies of \( B_i \) in \( C_i \)

Put \( C = P_N \)
The Partite Construction: Summary

- Ramsey Theorem:
  \[ K_n \rightarrow (K_{|B|})_{2|A|} \]
- Construct \( P_0 \)
- Enumerate by \( A_1, \ldots, A_N \) all possible projections of copies of \( A \) in \( P_0 \)
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- Construct \( P_1, \ldots, P_N \)
  - \( B_i \): partite system induced on \( P_{i-1} \) by all copies of all with projection to \( A_i \)
  - Partite lemma: \( C_i \rightarrow (B_i)_{A_i}^{A_i} \)
  - \( P_i \) is built repeated free amalgamation of \( P_i \) over all copies of \( B_i \) in \( C_i \)
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  - \( P_i \) is built repeated free amalgamation of \( P_i \) over all copies of \( B_i \) in \( C_i \)
- Put \( C = P_N \)
Lemma

Let $A$ be a structure s.t. $A = \{1, 2, \ldots, a\}$ and $B$ be an $A$-partite system.

Then there exists a $A$-partite system $C$ s.t.

$$C \rightarrow (B)^A_2.$$
The Partite Lemma

**Lemma**

Let $A$ be a structure s.t. $A = \{1, 2, \ldots, a\}$ and $B$ be an $A$-partite system.

Then there exists a $A$-partite system $C$ s.t.

$$C \rightarrow (B)_2^A.$$
The Partite Lemma

Proof by application of Hales-Jewett theorem

**Theorem (Hales-Jewett theorem)**

*For every finite alphabet $\Sigma$ there exists $N = HJ(\Sigma)$ so that for every 2-coloring of functions $h : \{1, 2, \ldots, N\} \to \Sigma$ there exists a monochromatic combinatorial line.*
The Partite Lemma

Proof by application of Hales-Jewett theorem

Theorem (Hales-Jewett theorem)

For every finite alphabet $\Sigma$ there exists $N = HJ(\Sigma)$ so that for every 2-coloring of functions $h : \{1, 2, \ldots, N\} \to \Sigma$ there exists a monochromatic combinatorial line.

Definition

For non-empty $\omega \subseteq \{1, 2, \ldots, N\}$ and $f : \{1, 2, \ldots, N\} \setminus \omega \to \Sigma$ combinatorial line $(\omega, f)$ is the set of all functions $f' : \{1, 2, \ldots, N\} \to \Sigma$ such that

$$f'(i) = \begin{cases} \text{constant for } i \in \omega, \\ f(i) \text{ otherwise.} \end{cases}$$
The Partite Lemma

Proof by application of Hales-Jewett theorem

\[ A = \begin{array}{c} 1 \\ 2 \end{array} \quad B = \begin{array}{c} X^1_B \\ X^2_B \end{array} \]

\[ \Sigma = \{ (a_x), (b_y), (b_z) \} \]

Build \( C \) so that functions \( h: \{1, 2, \ldots, N\} \to \Sigma \) correspond to copies of \( A \) and combinatorial lines to copies of \( B \):

Vertices in partition \( X_i \) of \( C \): Functions \( f: \{1, 2, \ldots, N\} \to X_i \) of \( B \).

Intended embedding \( B \to C \) corresponding the combinatorial line \((\omega, f)\):

\[ e_{\omega, f}(v_i) \{ v \text{ for } i \in \omega, \text{ vertex of } f(i) \text{ in the same partition as } v \text{ otherwise.} \]
The Partite Lemma

Proof by application of Hales-Jewett theorem

\[ A = \begin{array}{c}
1 \\
2 
\end{array} \quad B = \begin{array}{c}
\chi_B^1 \\
\chi_B^2 \\
x \\
y \\
z
\end{array} \]

\[ \Sigma = \{ (a_x), (b_y), (b_z) \} \] (alphabet describe all copies of A in B)
The Partite Lemma

Proof by application of Hales-Jewett theorem

\[
A = \begin{pmatrix}
1 \\
2
\end{pmatrix}
\quad B = \begin{pmatrix}
X^1_B \\
X^2_B
\end{pmatrix}
\]

- \( \Sigma = \{ (x, y, z) \} \) (alphabet describe all copies of \( A \) in \( B \))
- \( N = HJ(\Sigma) \)
The Partite Lemma

Proof by application of Hales-Jewett theorem

\[ A = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad B = \begin{bmatrix} a \\ b \\ x \\ y \\ z \end{bmatrix} \]

\[ \Sigma = \{ (a_x), (b_y), (b_z) \} \] (alphabet describe all copies of \( A \) in \( B \))

\[ N = HJ(\Sigma) \]

Build \( C \) so that functions \( h: \{1, 2, \ldots, N\} \to \Sigma \) correspond to copies of \( A \) and combinatorial lines to copies of \( B \):
The Partite Lemma

Proof by application of Hales-Jewett theorem

\[ \Sigma = \{ (a)_{X}, (b)_{Y}, (b)_{Z} \} \] (alphabet describe all copies of A in B)

\[ N = HJ(\Sigma) \]

Build C so that functions \( h : \{1, 2, \ldots, N\} \rightarrow \Sigma \) correspond to copies of A and combinatorial lines to copies of B:

- Vertices in partition \( X^{i}_{C} \): Functions \( f : \{1, 2, \ldots, N\} \rightarrow X^{i}_{B} \).
The Partite Lemma

Proof by application of Hales-Jewett theorem

\[ A = \begin{array}{c} 1 \\ 2 \end{array}, \quad B = \begin{array}{c} a \\ b \\ x \\ y \\ z \end{array} \]

\[ \Sigma = \{ (a, \chi), (b, y), (b, z) \} \] (alphabet describe all copies of A in B)

\[ N = HJ(\Sigma) \]

Build C so that functions \( h: \{1, 2, \ldots, N\} \to \Sigma \) correspond to copies of A and combinatorial lines to copies of B:

- Vertices in partition \( X_i^C \): Functions \( f: \{1, 2, \ldots, N\} \to X_B^i \).
- Intended embedding \( B \to C \) corresponding the combinatorial line \( (\omega, f) \):

\[ e_{\omega, f}(v)(i) \begin{cases} v \text{ for } i \in \omega, \\ \text{vertex of } f(i) \text{ in the same partition as } v \text{ otherwise.} \end{cases} \]
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Proof by application of Hales-Jewett theorem

\[ A = \begin{array}{c}
1 \\
2 
\end{array} \quad B = \begin{array}{c}
\begin{array}{c}
X^1_B \\
X^2_B
\end{array}
\end{array} \]

- \( \Sigma = \{ (a, x), (b, y), (b, z) \} \) (alphabet describe all copies of \( A \) in \( B \))
- \( N = HJ(\Sigma) \)
- Build \( C \) so that functions \( h : \{1, 2, \ldots, N\} \rightarrow \Sigma \) correspond to copies of \( A \) and combinatorial lines to copies of \( B \):
  - Vertices in partition \( X^i_C \): Functions \( f : \{1, 2, \ldots, N\} \rightarrow X^i_B \).
  - Intended embedding \( B \rightarrow C \) corresponding the combinatorial line \( (\omega, f) \):
    \[
e_{\omega, f}(v)(i) = \begin{cases} 
  v \text{ for } i \in \omega, \\
  \text{vertex of } f(i) \text{ in the same partition as } v \text{ otherwise.}
\end{cases}
\]
- Fact: It is possible to add tuples to relations as needed to make this work.
The Partite Lemma

Easy description of $C$:

$$A = \begin{array}{c}
1 \\
2 
\end{array} \quad B = \begin{array}{c}
X_1^1 \\
X_2^1 \\
X_2^2 \\
d \\
x \\
y \\
z 
\end{array}$$

Functions $f: \{1, 2, \ldots, N\} \to X_i$ such that all evaluation maps $g_i(f) = f(i)$ are homomorphisms from $C$ to $B$. 

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Ramsey Classes by Partite Construction I
The Partite Lemma

Easy description of $\mathbf{C}$:

Vertices in partition $\mathbf{X}_i^C$: Functions $f : \{1, 2, \ldots, N\} \to \mathbf{X}_i^B$

- Add as many tuples to relations as possible such that all the evaluation maps $g_i(f) = f(i)$ are homomorphisms from $\mathbf{C}$ to $\mathbf{B}$. 

J. Hubička
Ramsey Classes by Partite Construction I
Uses of the partite construction

Let class $\mathcal{K}$ be a class of structures satisfying given axioms. To show that $\mathcal{K}$ is Ramsey one can show that the partite construction preserve the axioms.
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Let class $\mathcal{K}$ be a class of structures satisfying given axioms. To show that $\mathcal{K}$ is Ramsey one can show that the partite construction preserve the axioms.

1. Nešetřil, Rödl, 1977: Classes with forbidden (amalgamation) irreducible structures
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3. Nešetřil, 2005: Metric spaces
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4. Nešetřil, 2010—: Classes with finitely many forbidden homomorphisms
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Uses of the partite construction

Let class $\mathcal{K}$ be a class of structures satisfying given axioms. To show that $\mathcal{K}$ is Ramsey one can show that the partite construction preserve the axioms.

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Let class $\mathcal{K}$ be a class of structures satisfying given axioms. To show that $\mathcal{K}$ is Ramsey one can show that the partite construction preserve the axioms.

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The Partite Construction

- Ramsey Theorem: \( K_n \rightarrow (K_{|B|})_{2}^{K_{|A|}} \)
- Construct \( P_0 \)
- Enumerate by \( A_1, \ldots, A_N \) all possible projections of copies of \( A \) in \( P_0 \)
- Construct \( P_1, \ldots, P_N \)
  - \( B_i \): partite system induced on \( P_{i-1} \) by all copies of all with projection to \( A_i \)
  - Partite lemma: \( C_i \rightarrow (B_i)_{2}^{A_i} \)
  - \( P_i \) is built repeated free amalgamation of \( P_i \) over all copies of \( B_i \) in \( C_i \)
- Put \( C = P_N \)
The **Induced Partite Construction**

- Nešetřil-Rödl Theorem: 
  \[ C_0 \rightarrow (B)_2^A \]
- Construct **$C_0$-partite** $P_0$
- Enumerate by $A_1, \ldots, A_N$ all possible projections of copies of $A$ in $P_0$
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  - $B_i$: partite system induced on $P_{i-1}$ by all copies of all with projection to $A_i$
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- Put \( C = P_N \)

If \( K \) is irreducible and \( A, B \) are \( K \)-free, then so is \( C \).
An exotic example

Bow-tie graph:
An exotic example

Bow-tie graph:

Amalgamation of two triangles must unify vertices.

Wrong!
Structure of bow-tie-free graphs

Definition
A chimney is a graph created by gluing multiple triangles over one edge.

Definition
A graph is good if every vertex is either in a chimney or $K_4$. 

Edges in no triangles  Edges in 1 triangle  Edges in 2+ triangles
Definition

**Chimney** is a graph created by gluing multiple triangles over one edge.

Definition

Graph is **good** if every vertex is either in a chimney or $K_4$. 
Structure of bow-tie-free graphs

Graph is good if every vertex is either in a chimney or $K_{4}$.

Every bowtie-free graph $G$ is a subgraph of some good graph $G'$.

For every good graph $G = (V, E)$ the graph $G_{\Delta} = (V, E_{\Delta})$ ($E_{\Delta}$ are edges in triangles) is a disjoint union of copies of chimneys and $K_{4}$.

Closure of a vertex $v$ = all endpoints of red edges contained in triangles containing $v$.

Lemma
Bow-tie-free graphs have free amalgamation over closed structures.
Graph is **good** if every vertex is either in a chimney or $K_4$. 

**Structure of bow-tie-free graphs**

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Closure of a vertex $v =$ all endpoints of red edges contained in triangles containing $v$.

**Lemma**

*Bow-tie-free graphs have free amalgamation over closed structures.*
3 types of vertices and their closures:

- [Diagram of a triangle with one vertex highlighted and two others connected by a path]
- [Diagram of a complete graph with all vertices connected by edges]
3 types of vertices and their closures:

To describe lift of bowtie graphs we only need to forbid all triangles except for B-B-R and R-R-R.
3 types of vertices and their closures:

To describe lift of bowtie graphs we only need to forbid all triangles except for B-B-R and R-R-R.

Theorem (H., Nešetřil, 2014)

The class of graphs not containing bow-tie as non-induced subgraph have Ramsey lift.
Unary closure description $C$ is a set of pairs $(R^U, R^B)$ where $R^U$ is unary relation and $R^B$ is binary relation.

We say that structure $A$ is $C$-closed if for every pair $(R^U, R^B)$ the $B$-outdegree of every vertex of $A$ that is in $U$ is 1.

**Theorem (H., Nešetřil, 2015)**

Let $\mathcal{E}$ be a family of complete ordered structures and $\mathcal{U}$ an unary closure description. Then the class of all $C$-closed structures in $\text{Forb}_E(\mathcal{E})$ has Ramsey lift.
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Let $\mathcal{E}$ be a family of complete ordered structures and $\mathcal{U}$ an unary closure description. Then the class of all $\mathcal{C}$-closed structures in $\text{Forb}_E(\mathcal{E})$ has Ramsey lift.

All Cherlin Shelah Shi classes with unary closure can be described this way!
Nešetřil-Rödl Theorem: $C_0 \rightarrow (B)^A_2$

Construct $C_0$-partite $P_0$

Enumerate by $A_1, \ldots, A_N$ all possible projections of copies of $A$ in $P_0$

Construct $C_0$-partite $P_1, \ldots, P_N$:

$B_i$ is partite system induced on $P_i - 1$ by all copies of all with projection to $A_i$

Partite lemma: $C_i \rightarrow (B_i)^A$

$P_i$ is built by repeated free amalgamation of $P_i$ over all copies of $B_i$ in $C_i$

Put $C = P_N$

$B_i$ are $C$-closed. Only potential problem is the partite construction.
Nešetřil-Rödl Theorem: \( C_0 \rightarrow (B)_2^A \)

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The Induced Partite Construction with unary closure

- Nešetřil-Rödl Theorem: \( C_0 \longrightarrow (B)^A_2 \)
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J. Hubička
Ramsey Classes by Partite Construction I
The Induced Partite Construction with unary closure

- Nešetřil-Rödl Theorem: $C_0 \rightarrow (B)^A_2$
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- Put $C = P_N$

$A, B, B_i$ are $C$-closed. Only potential problem is the partite construction.
The Partite Lemma

Easy description of $\mathbf{C}$:

\[
\begin{align*}
A &= \begin{array}{c}
\bullet \\
\bullet
\end{array} \\
B &= \begin{array}{cccc}
\bullet & a & b \\
\bullet & x & y & z
\end{array}
\end{align*}
\]
The Partite Lemma

Easy description of $C$:

$A = \{x, y, z\}$

$B = \{a, b\}$

Vertices in partition $X^i_B$: Functions $f : \{1, 2, \ldots, N\} \rightarrow X^i_B$

Add as many tuples to relations as possible such that all the evaluation maps $g_i(f) = f(i)$ are homomorphisms from $C$ to $B$. 

J. Hubička

Ramsey Classes by Partite Construction I
The Partite Lemma

Easy description of $\mathbf{C}$:

$\mathbf{A} = \mathbf{B} = \{a, b, x, y, z\}$

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out-degree 1 is preserved:
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\mathbf{B} &= \begin{array}{c}
\bigcirc \\
\bigcirc \\
\bigcirc
\end{array}
\end{align*}
\]

- Vertices in partition $\mathbf{X}_i^C$: Functions $f : \{1, 2, \ldots, N\} \rightarrow \mathbf{X}_B^i$
- Add as many tuples to relations as possible such that all the evaluation maps $g_i(f) = f(i)$ are homomorphisms from $\mathbf{C}$ to $\mathbf{B}$.

out-degree 1 is preserved:

\[
f(1) = x, f(2) = y, f(3) = z, f(4) = x, \ldots
\]
The Partite Lemma

Easy description of $C$:

Vertices in partition $X_i^C$: Functions $f : \{1, 2, \ldots, N\} \rightarrow X_i^B$

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If there is edge from $f$ to $f'$ then:

$$f'(1) =$$
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Easy description of $C$:

![Graph showing vertices and edges]

- Vertices in partition $X^i_C$: Functions $f : \{1, 2, \ldots, N\} \rightarrow X^i_B$
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The Partite Lemma

Easy description of $C$:

$A = \begin{array}{c}
\bullet \\
\bullet
\end{array}$

$B = \begin{array}{c}
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\bullet \ b \\
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If there is edge from $f$ to $f'$ then:

$$f'(1) = a, f'(2) = b,$$
The Partite Lemma

Easy description of $\mathbf{C}$:

$A = \begin{array}{c} \bullet \\ \textcolor{red}{\bullet} \end{array}$  $B = \begin{array}{c} \bullet \textcolor{red}{a} \\ \bullet \textcolor{red}{b} \\ \bullet \setminus \\ \bullet \textcolor{red}{x} \\ \bullet \textcolor{red}{y} \\ \bullet \textcolor{red}{z} \end{array}$

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out-degree 1 is preserved:

$$f(1) = x, f(2) = y, f(3) = z, f(4) = x, \ldots$$

If there is edge from $f$ to $f'$ then:

$$f'(1) = a, f'(2) = b, f'(3) = b, f'(4) = a, \ldots$$
Map of Ramsey Classes

boolean algebras
interval graphs
permutations
cyclic orders
unions of complete graphs
linear orders
restricted

metric spaces
acyclic graphs
partial orders
$K_n$-free graphs
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Ramsey Classes by Partite Construction I
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Generalizing Partite Construction to non-unary closure

Nešetřil-Rödl Theorem: $C_0 \rightarrow (B_2)$

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Unify vertices to preserve out-degree 1 of non-unary closure edges
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Example: \( QQ \)

**Definition**

Denote by \( QQ \) the structure with binary relation \( \leq \) and ternary relation \( \prec \) with the following properties

1. relation \( \leq \) forms the generic linear order

2. for every vertex \( a \in QQ \) the relation \( \{(b, c) : (a, b, c) \in \prec\} \) forms the generic linear order on \( QQ \setminus \{a\} \) that is free to \( \leq \).
Example: $\mathbb{QQ}$

### Definition

Denote by $\mathbb{QQ}$ the structure with binary relation $\leq$ and ternary relation $\prec$ with the following properties

1. relation $\leq$ forms the generic linear order
2. for every vertex $a \in \mathbb{QQ}$ the relation $\{(b, c) : (a, b, c) \in \prec\}$ forms the generic linear order on $\mathbb{QQ} \setminus \{a\}$ that is free to $\leq$.

Use alternative representation with binary relations and closures to show that the age of $\mathbb{QQ}$ is a Ramsey class:
Example: $\mathbb{QQ}$

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Example: QQ

**Definition**

Denote by $\mathbb{Q} \mathbb{Q}$ the structure with binary relation $\leq$ and ternary relation $\prec$ with the following properties:

1. The relation $\leq$ forms the generic linear order.
2. For every vertex $a \in \mathbb{Q} \mathbb{Q}$, the relation $\{(b, c) : (a, b, c) \in \prec\}$ forms the generic linear order on $\mathbb{Q} \mathbb{Q} \setminus \{a\}$ that is free to $\leq$.

Use alternative representation with binary relations and closures to show that the age of $\mathbb{Q} \mathbb{Q}$ is a Ramsey class:

![Graph Diagram]
Example: $QQ$

**Definition**

Denote by $QQ$ the structure with binary relation $\leq$ and ternary relation $\prec$ with the following properties:

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Use alternative representation with binary relations and closures to show that the age of $QQ$ is a Ramsey class:

![Diagram](image-url)
The End

THANK YOU!