Big Ramsey degrees of the 3-uniform hypergraph

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Ramsey Theorem

Theorem (Finite Ramsey Theorem, 1930)

∀n,p,k ≥ 1 ∃N : N → (n)_k,1.

N → (n)_k,t : For every partition of (1,2,...,N) into k classes (colours) there exists X ⊆ {1,2,...,N}, |X| = n such that (X)_p belongs to at most t partitions (if t = 1 it is monochromatic)

For p = 2, n = 3, k = 2 put N = 6
Ramsey theorem for finite structures

Denote by $\vec{H}_l$ the class of all finite $l$-uniform hypergraphs endowed with linear order on vertices.

**Theorem (Nešetřil-Rödl, 1977; Abramson-Harrington, 1978)**

$$\forall l \geq 2, A, B \in \vec{H}_l \exists C \in \vec{H}_l : C \rightarrow (B)^A_{2,1}.$$
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\forall _{l \geq 2, A, B \in \mathcal{H}_l} \exists _{C \in \mathcal{H}_l} : C \rightarrow (B)^A_{2,1}.
\]

$\binom{B}{A}$ is the set of all induced sub-hypergraphs of $B$ isomorphic to $A$.

$C \rightarrow (B)^A_{k,t}$: For every $k$-colouring of $\binom{C}{A}$ there exists $\tilde{B} \in \binom{C}{B}$ such that $\binom{\tilde{B}}{A}$ has at most $t$ colours.
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Order is necessary

Vertices of $C$ can be linearly ordered and copies of $A$ colored:

- red if middle vertex appears first.
- blue otherwise.

Every ordering of 5-cycle contains minimal and maximal element. Consequently, every 5-cycle in $C$ contains both blue and red copies of $A$. 
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Hypergraphs have finite small Ramsey degree

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$$\forall l \geq 2, k \geq 2, A, B \in \mathcal{H}_l \exists C \in \mathcal{H}_l : C \rightarrow (B)_k, t(A).$$

where $t(A)$, the small Ramsey degree of $A$ in $\mathcal{H}_l$, is the number of non-isomorphic ordering of vertices of $A$. 
Definition

A class $C$ of finite $L$-structures is **Ramsey** iff $\forall_{A,B \in C} \exists_{C \in C} : C \rightarrow (B)^A_{2,1}$. 
Ramsey classes

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The class of all finite linear orders is a Ramsey class.
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For every relational language $L$, $Rel(L)$ is a Ramsey class.
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Example (Partial orders — Nešetřil-Rödl, 84; Paoli-Trotter-Walker, 85)

The class of all finite partial orders with linear extension is Ramsey.
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Example (Models — H.-Nešetřil, 2016)

For every language $L$, $\overrightarrow{Mod}(L)$ is a Ramsey class.
Gower's Ramsey Theorem

Graham Rotschild Theorem: Parametric words

Milliken tree theorem: C-relations

Ramsey’s theorem: rationals
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Product arguments
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Graham Rotschild Theorem: Parametric words → Boolean algebras

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Free amalgamation classes
→ Rel(L)

Product arguments  Interpretations  Adding unary functions  Partite construction
Gower's Ramsey Theorem

Graham Rotschild Theorem: Parametric words

Dual structural Ramsey theorem

Boolean algebras

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Partial Steiner systems

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Metric spaces

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Big Ramsey Degrees

**Theorem (Infinite Ramsey Theorem)**

\[ \forall p, k \geq 1 \mathbb{N} \rightarrow (\mathbb{N})_{k,1}^p. \]

By \( \mathbb{N} \) we denote the order of integers (without arithmetic operations on them).
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Theorem (Devlin, 1979)

\[ \forall p, k \geq 1 \mathbb{Q} \rightarrow (\mathbb{Q})^{p}_{k,T(k)}. \]

*For certain finite \( T(k) \). \( T(k) \) is the big Ramsey degree of \( k \) tuple in \( \mathbb{Q} \).*
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\[ T(k) = \tan^{(2k-1)}(0). \]

tan\(^{(2k-1)}(0)\) is the \((2k - 1)\)st derivative of the tangent evaluated at 0.
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Ramseyness is an application of Milliken’s tree theorem on binary tree.
Rich colouring of $\mathbb{Q}$
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$\leq x_0$ $\geq x_0$
Rich colouring of $\mathbb{Q}$
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$x_0$ $x_1$
Rich colouring of \( \mathbb{Q} \)
Rich colouring of $\mathbb{Q}$

- $\leq x_0$
- $\geq x_0, \leq x_1$
- $\geq x_1$

Diagram:
- $x_2$
- $x_0$
- $x_1$
Rich colouring of $\mathbb{Q}$

The image shows a diagram with three nodes labeled $x_0$, $x_1$, and $x_2$. The nodes are connected in a tree-like structure, with $x_0$ as the root, $x_1$ and $x_2$ as intermediate nodes, and the edges indicating the meet closure in the tree.
Rich colouring of $\mathbb{Q}$

- $x_0$
- $x_1$
- $x_2$
- $x_3$
- $x_4$
- $x_5$
- $x_6$

The colour of a $k$-tuple corresponds to the shape of the meet closure in the tree.
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Colour of $k$-tuple = shape of meet closure in the tree
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Big Ramsey degrees of $\mathbb{Q}$ as trees

We describe tree using two orders.

1. $\leq$ is the order of rationals
2. $\preceq$ is the well-order fixed by enumeration

$T(X, \leq, \preceq)$ is the tree built by previous procedure for set $(X, \leq)$ executed in order given by $\preceq$. (A binary search tree used in computer science.)
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**Definition**

We say that $(X, \leq)$ and $(X, \preceq)$ is a compatible pair of orders of $X$ if and only if $\leq$ and $\preceq$ are linear orders and every vertex of $X$ is either leaf or has 2 sons.

$T(3) = 16$
Big Ramsey degrees of \( \mathbb{Q} \) as trees

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The big Ramsey degree of \( n \) tuples is precisely given by number of compatible pair of orders on an \( n \)-tuple.
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\[
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\]
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$R$

$\nu_0$
Rich colouring of the countable random graph

\[ R \]

\[ v_0 \]
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\[ R \]

\[ v_0 \quad v_1 \quad v_2 \]

Colour of a subgraph = shape of meet closure in the tree

Given well order \( \preceq \) (bottom-up) one can define dense order by listing the tree from left to right.
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Sauer’s theorem

Big Ramsey degrees of graphs are given by Devlin’s trees annotated by graph edges.
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\[
\begin{align*}
\text{NO:} & & \quad \text{NO:} \\
\end{align*}
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Definition

Let \((\leq, \preceq)\) be a pair of compatible orders of a set \(V'\), let \(V\) be the set of leaf vertices of \(T(V', \leq, \preceq)\), and let \(G = (V, E)\) be a graph. We say that \(G\) is compatible with \(T(V', \leq, \preceq)\) if, for every triple \(a, b, c \in V\) of distinct vertices satisfying \(c \preceq (a \land b)\), we have \(\{a, c\} \in E\) if and only if \(\{b, c\} \in E\).
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Theorem (Sauer 2006)

Let $R$ be the random graph.

\[ \forall_{\text{finite graph } G \geq 1} R \rightarrow (R)^G_{k,T(H)}. \]

Where $T(G)$ is the Ramsey degree of a graph Ramsey degree of a graph $G = ([n], E)$ in $R$. 

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Where \(T(G)\) is the Ramsey degree of a graph Ramsey degree of a graph \(G = ([n], E)\) in \(R\).

\(T(G)\) is the number of non-isomorphic structures \(([2n - 1], E, \leq, \preceq)\) where \((\leq, \preceq)\) is a pair of compatible linear orders of \([2n - 1]\) and \(G\) is compatible with \(T([2n - 1], \leq, \preceq)\).

Proved again by the application of Milliken tree theorem on binary tree.
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$$H_3$$
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Colour of a subgraph = shape of meet closure in the tree

**Problem**: Ramsey theorem for this type of tree does not hold
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Year later we observed that neighbourhood of a vertex is the Random graph!
Rich colouring of the countable random 3-uniform hyper-graph

Colour of a subgraph = shape of meet closure in both trees

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Year later we observed that neighbourhood of a vertex is the Random graph!
Big Ramsey degrees of 3-uniform hypergraphs are pairs of trees
Big Ramsey degrees of 3-uniform hypergraphs as product trees

Definition

Let $(\leq, \preceq)$ be a pair of compatible orders of a set $V'$, let $V$ be the set of leaf vertices of $T(V', \leq, \preceq)$, and let $G = (V, E)$ be a 3-uniform hypergraph. We say that $G$ is compatible with $T(V', \leq, \preceq)$ if for every 4-tuple $a, b, c, d$ of distinct vertices of $V$ satisfying $d \preceq c \preceq (a \wedge b)$ we have $\{a, c, d\} \in E$ if and only if $\{b, c, d\} \in E$.

Given a tree $T(V^0, \leq, \preceq)$ and a compatible 3-uniform hypergraph $G = (V, E)$, we define the neighbourhood graph of $G$ with respect to $T(V^0, \leq, \preceq)$ as the graph $G^1 = (V^1, E^1)$ constructed as follows:

- $V^1$ consists of all pairs $(a, b)$ such that $a \in V$ (by compatibility $V \subseteq V^0$) and $b \in V^0$, $a \prec b$ and there is no $c \in V^0, c \sqsubset b$ such that $a \prec c \prec b$.
- $\{(a, b), (c, d)\} \in E^1$ for $a \preceq c$ if there exists $e \sqsupseteq d$ such that $\{a, c, e\} \in E$. (This is well defined because of the compatibility of $T(V^0, \leq, \preceq)$ and $G$.)

For $(a, b) \in V'$, we define its projection $\pi : V \times V^0 \to V$ by putting $\pi((a, b)) = a$.

Definition

The tuple $(V^0, V^1, \preceq, \leq^0, \leq^1)$ is compatible with the 3-uniform hypergraph $G = (V, E)$ iff:

- $V^0 \cap V^1 = \emptyset$,
- $(\leq^0, \preceq \upharpoonright V^0)$ is a compatible pair of orders of $V^0$ and $T(V^0, \leq^0, \preceq \upharpoonright V^0)$ is compatible with $G$,
- $(\leq^1, \preceq \upharpoonright V^1)$ is a compatible pair of orders of $V^1$ and $T(V^1, \leq^1, \preceq \upharpoonright V^1)$ is compatible with the neighbourhood graph $G^1 = (V^1, E^1)$ of $G$ with respect to $T(V^0, \leq^0, \preceq \upharpoonright V^0)$,
- $\preceq$ is a well pre-order which satisfies $a \neq b, a \preceq b, b \preceq a \Rightarrow \pi(a) = \pi(b)$, and both projections are defined. Moreover, whenever $\pi(a)$ and $\pi(b)$ are defined, $\pi(a) \preceq \pi(b) \Rightarrow a \preceq b$. Finally, for $(a, b), (c, d) \in V^1$, we have $((a, b) \wedge (c, d)) \prec (b \wedge d)$. 
Structural Ramsey Theory

Big Ramsey Degrees of $\mathbb{Q}$

Random graph

Random hypergraph

Big Ramsey degrees of 3-uniform hypergraphs are finite

Theorem (Balko, Chodounský, H., Konečný 2019+)

Let $H_3$ be the random 3-uniform hypergraph.

\[ \forall \text{finite hypergraph } G, k \geq 1 \quad H_3 \rightarrow (H_3)_k, T(G) \cdot \]

The big Ramsey degree of a 3-uniform hypergraph $G = ([n], \mathcal{E})$ in $H_3$ is the number of non-isomorphic structures $([2n - 1] \cup V^1, \preceq, \preceq^0, \preceq^1, \mathcal{E}, \mathcal{P})$ such that $([2n - 1], V^1, \preceq, \preceq^0, \preceq^1)$ is compatible with $\mathcal{E}$, $\preceq|_{[2n-1]}$ is a linear order and $\mathcal{P}$ consists of all triples $\{a, b, (a, b)\}$ such that $(a, b)$ is a vertex of the neighbourhood graph $G^1$. 
Big Ramsey degrees of 3-uniform hypergraphs are finite

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The **big Ramsey degree** of a 3-uniform hypergraph $G = ([n], E)$ in $H_3$ is the number of non-isomorphic structures $([2n-1] \cup V^1, \leq, \leq^0, \leq^1, E, P)$ such that $([2n-1], V^1, \leq, \leq^0, \leq^1)$ is compatible with $E$, $\leq|_{[2n-1]}$ is a linear order and $P$ consists of all triples $\{a, b, (a, b)\}$ such that $(a, b)$ is a vertex of the neighbourhood graph $G^1$.

Big Ramsey degree bounds are known for the following:

1. **Order of integers** (Ramsey 1930)
2. **Order of rationals** (Devlin 1979)
3. **Random graph** (Sauer 2006)
4. **Dense local order** (Laflamme, Nguyen Van Thé, Sauer 2010)
5. **Ultrametric spaces** (Nguyen Van Thé 2010)
6. **Universal $K_k$-free graphs for $k \geq 2$** (Dobrinen 2018+)
7. **Structures with unary functions only** (H., Nešetřil, 2019)
8. **Some structures with equivalence relations** (Howe 2019+, H.+)
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Kechris, Pestov and Todorcevic linked big Ramsey degrees to topological dynamics. This was recently developed by Andy Zucker to the notion of Big Ramsey structures.
Thank you for the attention


