On Big Ramsey degrees of universal ω-edge-labeled hypergraphs

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Prague

Joint work with Matěj Konečný, Stevo Todorčević and Andy Zucker

EuroComb, 2025

The Random (Rado) graph **R**

Theorem (Erdős–Rényi, 1963)

There exists a countable graph **R** *(the Random or Rado graph) with the following property. If a countable graph is chosen at random (by selecting edges independently with probability $\frac{1}{2}$) then with probability 1, the resulting graph is isomorphic to* **R**.

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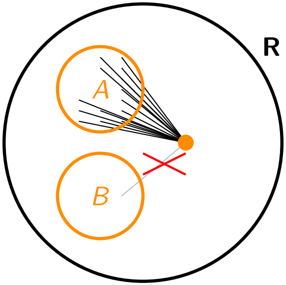

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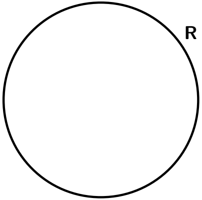

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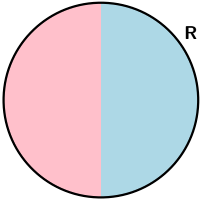

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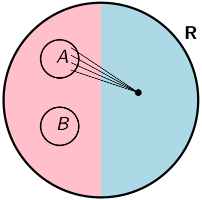

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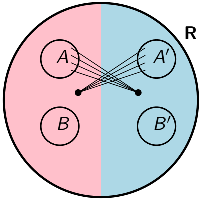

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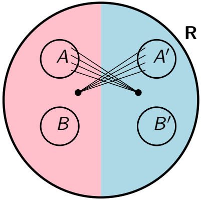

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4. A contradiction with EP of **R** for $A \cup A'$ and $B \cup B'$.

Colouring subgraphs of $\mathbf{R}$

Theorem (Erdős–Hajnal–Pósa, 1974)

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Theorem (Sauer, 2006)

Let $\mathcal{G}$ *be the class of all finite graphs.*

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By $\binom{\mathbf{B}}{\mathbf{A}}$ we denote the set of all embeddings of **A** to **B**.

Definition (Leeb's generalization of the Erdős–Rado partition arrow)

$\mathbf{C} \longrightarrow (\mathbf{B})^{\mathbf{A}}_{k,t}$ means:

For every k-colouring of $\binom{\mathbf{C}}{\mathbf{A}}$ there exists $f \in \binom{\mathbf{C}}{\mathbf{B}}$ such that $\binom{f(\mathbf{B})}{\mathbf{A}}$ has at most t colours.

Minimal possible value of $T(\mathbf{A})$ is a big Ramsey degree of **A** in **R**.

Colouring embeddings (copies) of hypergraphs

Definition

Given a countable set L of *labels*, an L-edge-labeled u-uniform hypergraph (or simply an edge-labeled hypergraph) is a pair $\mathbf{A} = (A, e_{\mathbf{A}})$, where $e_{\mathbf{A}}$ is a function $e_{\mathbf{A}} \colon \binom{A}{u} \to L$.

Given L-edge-labeled u-uniform hypergraphs $\mathbf{A} = (A, e_{\mathbf{A}})$ and $\mathbf{B} = (B, e_{\mathbf{B}})$, an embedding $f \colon \mathbf{A} \to \mathbf{B}$ is an injective function $f \colon A \to B$ which preserves the labels.

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Definition (Big Ramsey degree)

Given (countably infinite) edge-labelled hypergraph $\mathbf{H}$ and finite edge-labelled hypergraph $\mathbf{A}$, the big Ramsey degree of $\mathbf{A}$ in $\mathbf{H}$ is the smallest T such that

$$\mathbf{H} \longrightarrow (\mathbf{H})^{\mathbf{A}}_{T+1,T}$$

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Today: If L is infinite ($L = \omega$) the big Ramsey degrees are no longer finite.

Some results on finiteness of big Ramsey degrees

Finiteness of big Ramsey degrees of the order of rationals was shown by Laver in 1969 and characterised by Devlin in 1979. The topic was revitalized in 2005 by Kechris, Pestov and Todorčevič.

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4. Dobrinen (2020): Big Ramsey degrees of universal homogeneous triangle-free graphs are finite
5. Dobrinen (2023): Big Ramsey degrees of universal homogeneous K_k-free graphs are finite for every $k \geq 3$.
6. Zucker (2022): Big Ramsey degrees of Fraïssé limits of free amalgamation classes in binary language with finitely many forbidden substructures are finite.
7. J.H. (2025): Big Ramsey degrees of partial orders and metric spaces are finite.
8. Balko, Chodounský, Dobrinen, J.H., Konečný, Nešetřil, Vena, Zucker (2021): Big Ramsey degrees of structures described by induced cycles are finite.
9. Balko, Chodounský, Dobrinen, J.H., Konečný, Vena, Zucker (2024): Characterisation of big Ramsey degrees of Fraïssé limits of free amalgamation classes in binary language with finitely many constraints.
10. Bice, de Rancourt, J.H., Konečný: metric big Ramsey degrees of ℓ_∞ and the Urysohn sphere,

Known negative results

1. Urysohn (metric) space (folklore?)
2. Urysohn sphere (folclore?)
3. Real numbers, topological copies of the order of rationals (Todorčević, 1987)
4. Henson graphs with finitely many forbidden cliques (El-Zahar, Sauer, 1993)
5. Pseudotree (Chodounský, Monroe, Weinert, 2025+)
6. Boolean algebras (Bartošová, Chodounský, Csima, H., Konečný, Lakerdas-Gayle, Unger, Zucker, 2025+)

Our contribution

Denote by $\mathbf{R}^u_\omega$ the ω-edge-labelled u-uniform hypergraph.

Theorem (H., Konečný, Todorčević, Zucker)

Let $u > 1$ be finite and let $\mathbf{A}$ be any ω-edge-labeled u-uniform hypergraph with 2 vertices. Then for no finite T satisfies

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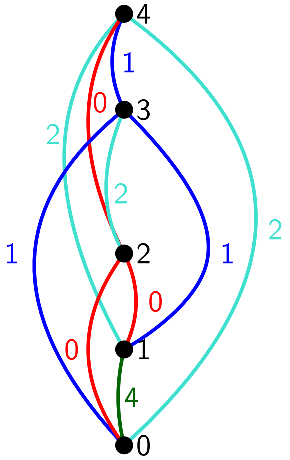
Colouring of $\binom{\mathbf{R}^u_\omega}{\mathbf{A}}$ is persistent if every copy of $\mathbf{R}^u_\omega$ touches every colour. Given $u \geq 2$ and $T > 1$ we give such persistent colorng of $\binom{\mathbf{R}^u_\omega}{\mathbf{A}}$ with $T+1$ colors.

For simplicity, let's consider only the ω-edge-labelled graph $\mathbf{R}^2_\omega$.

Persistent colouring of edges



An initial segment of infinie ω-edge-labelled graph $\mathbf{R}^2_\omega$

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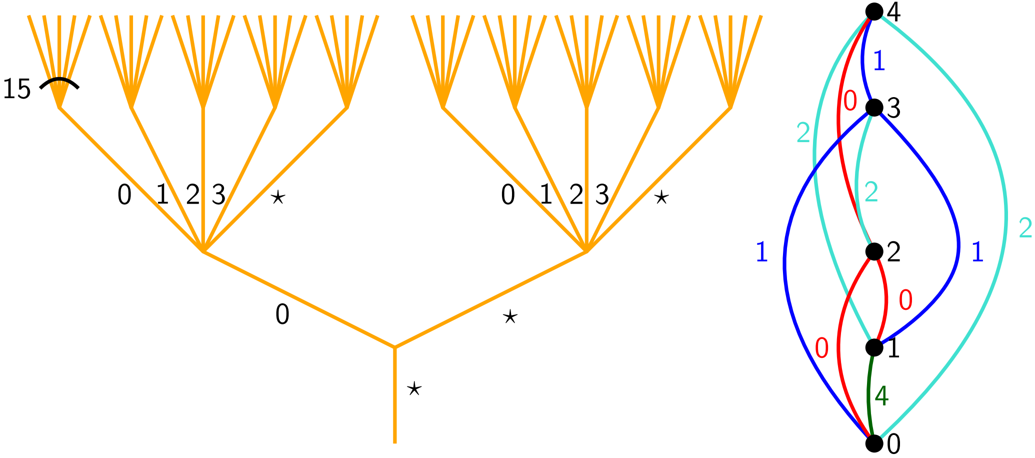

An infinite finitely-branching tree where the number of sons of a node x exceeds the number of nodes on the level of x and below. Sons are enumerated; last son has label $\star$

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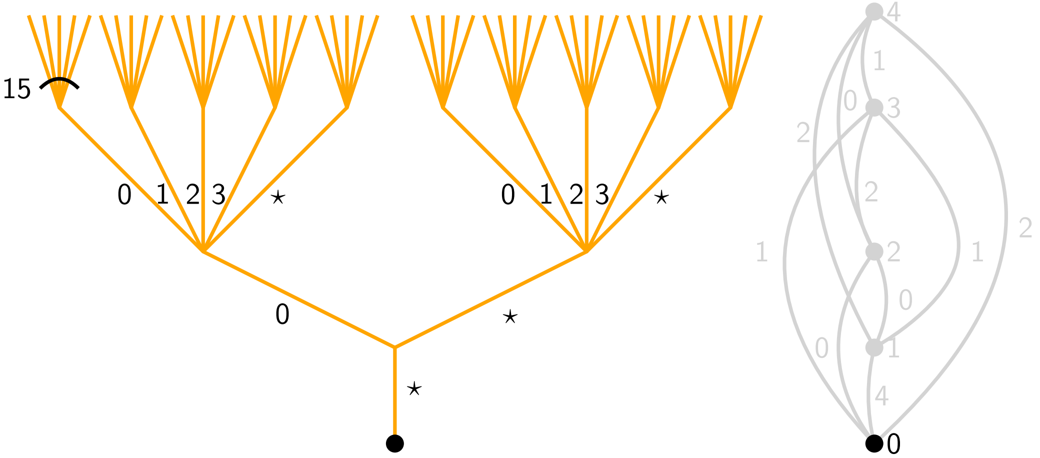

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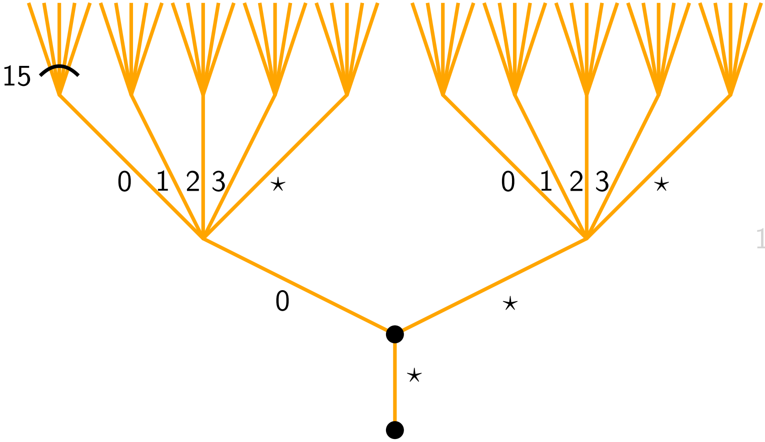

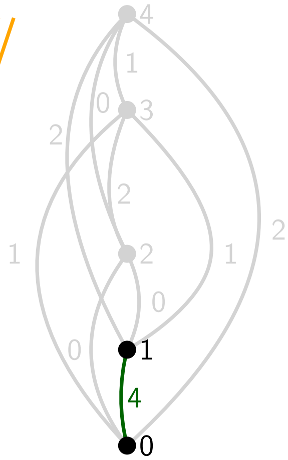

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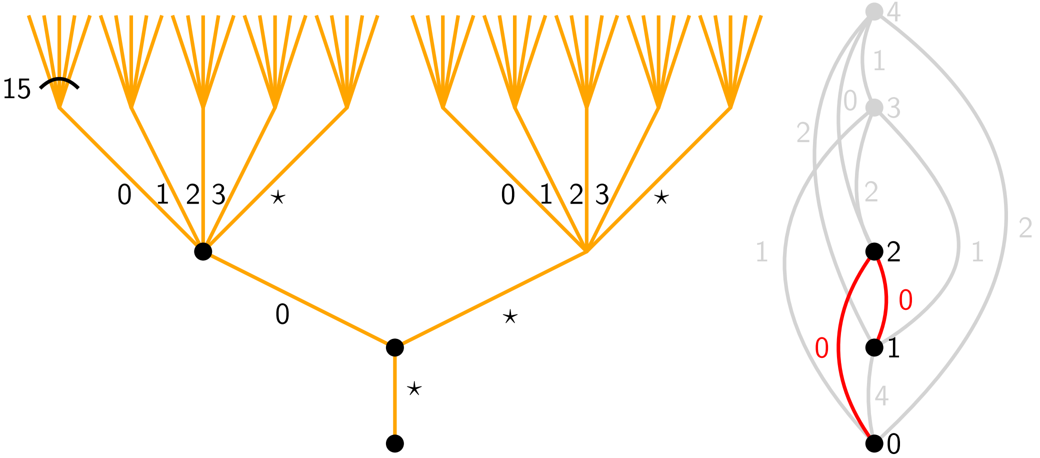

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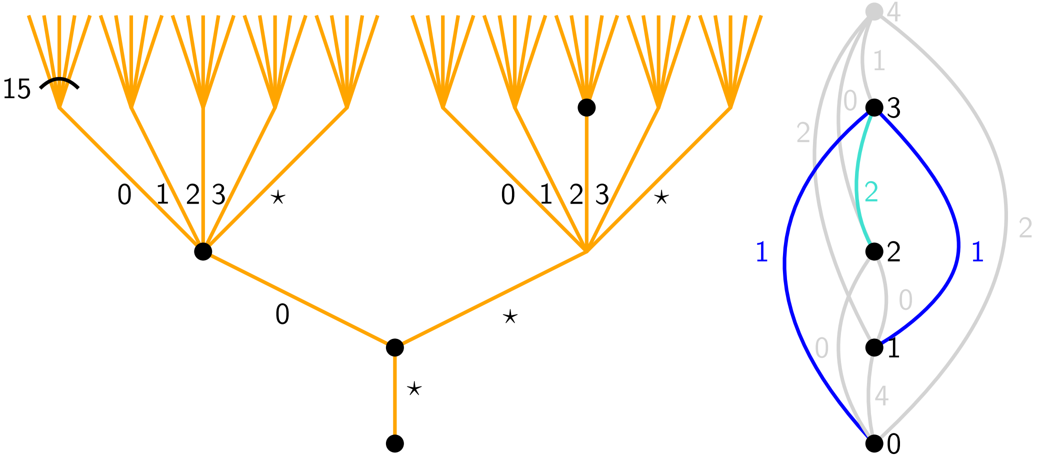

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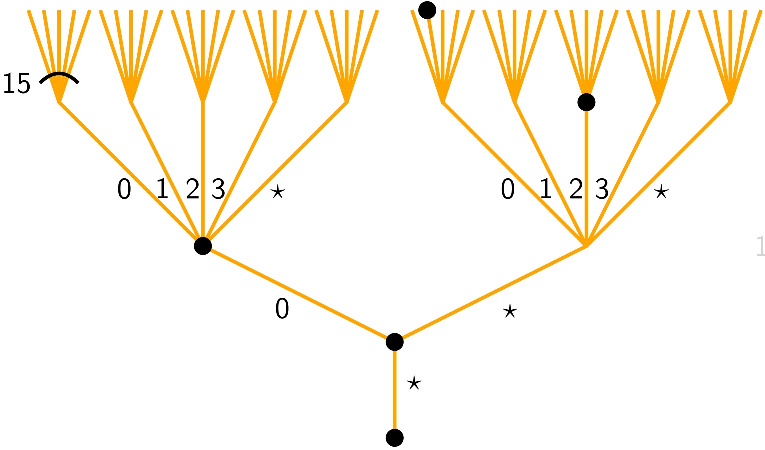

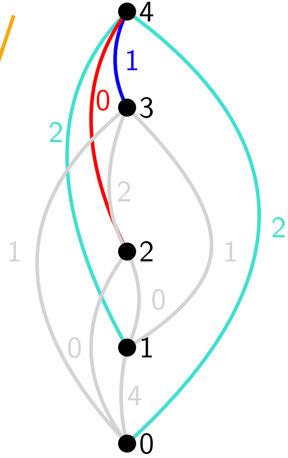

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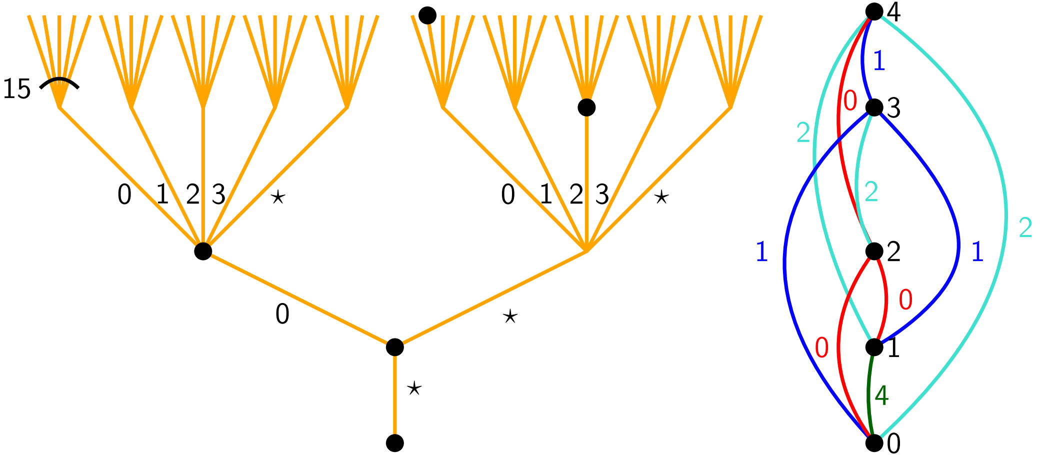

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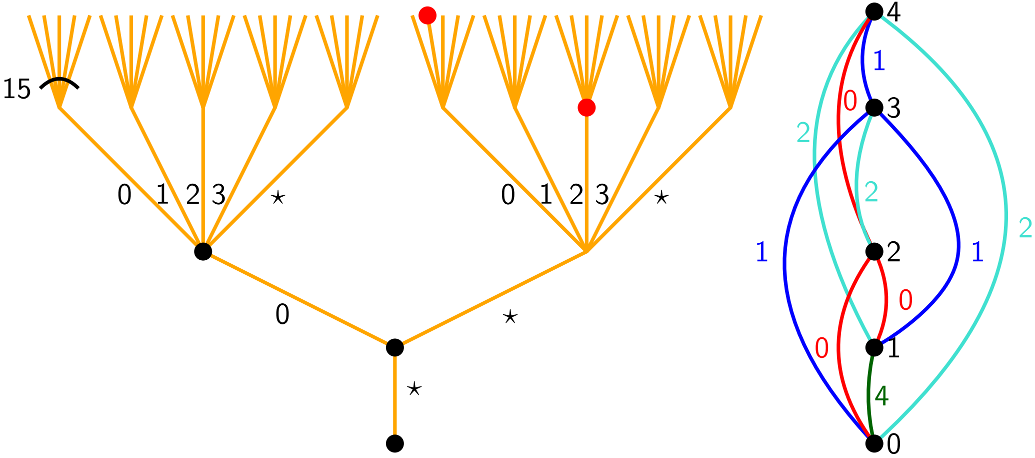

Every pair of nodes determines a unique meet (nearest common ancestor)

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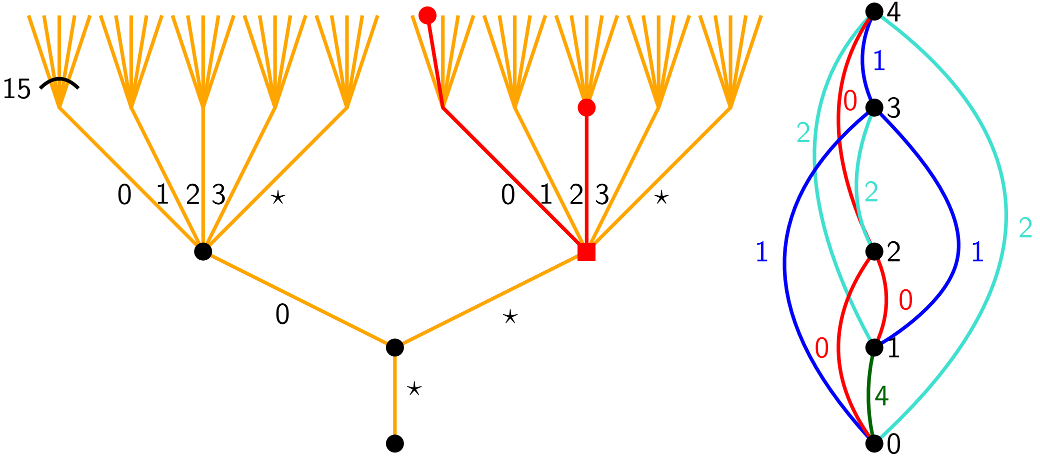

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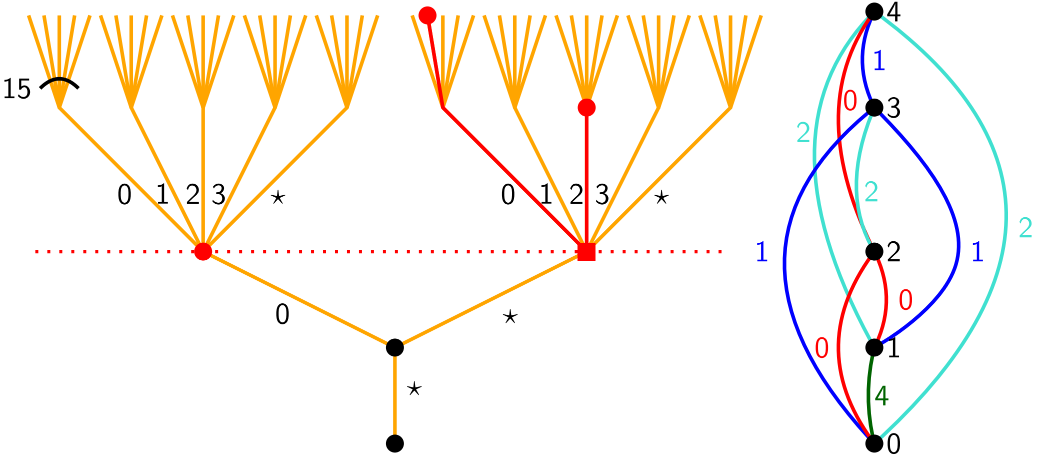
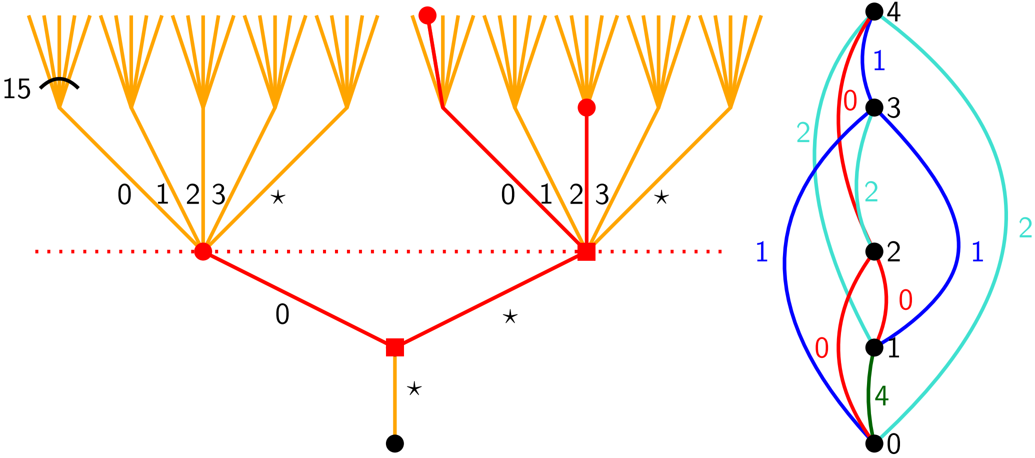


Every nearest common ancestor determines a unique coding node

Persistent colouring of edges



Meet and coding node determine another meet

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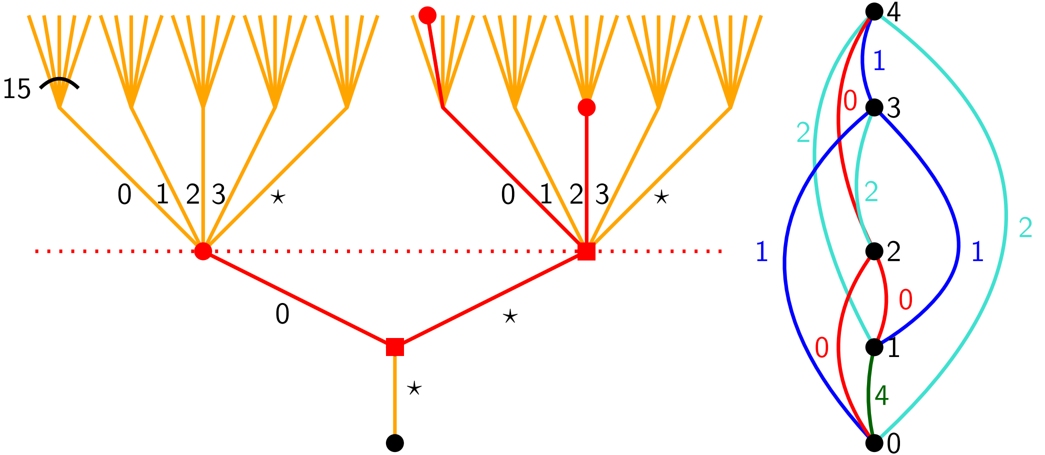
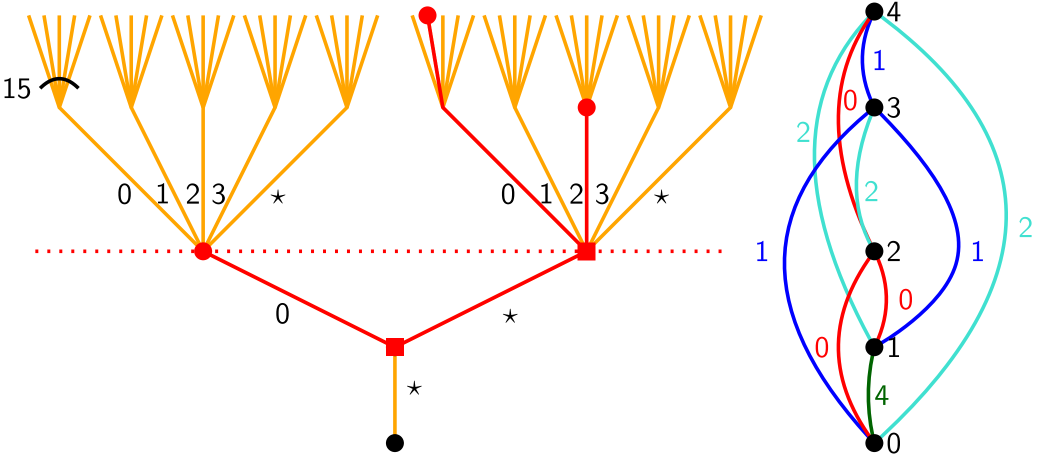


Definition

height(v,w) is the number of meets which can be reached from coding nodes corresponding to vertices *v*, *w* by this procedure.

Persistent colouring of edges



Theorem

For every embedding $\varphi : \mathbf{R}^2_\omega \to \mathbf{R}^2_\omega$ *there exists an integer* m *such that for every* $n > m$ *there exists an edge* (v, w) *of label* 0 *in the image of* φ *of height* n*.*

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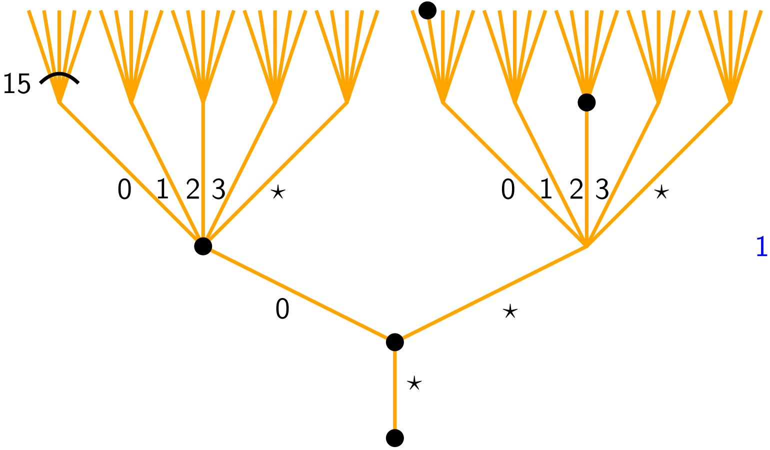

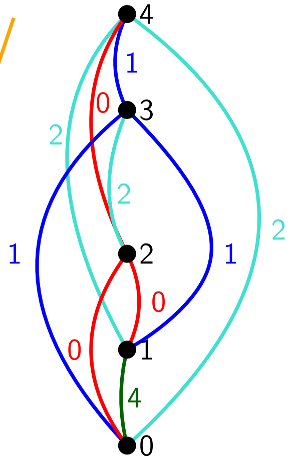

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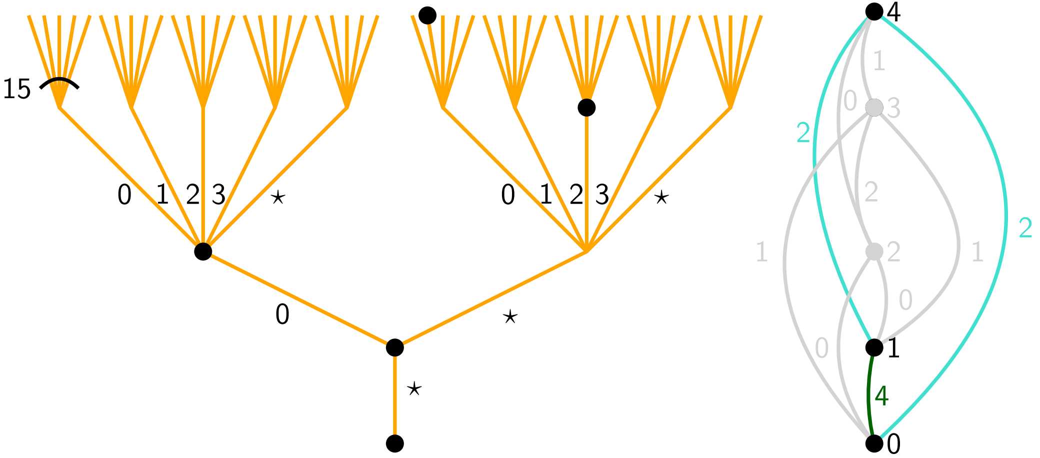

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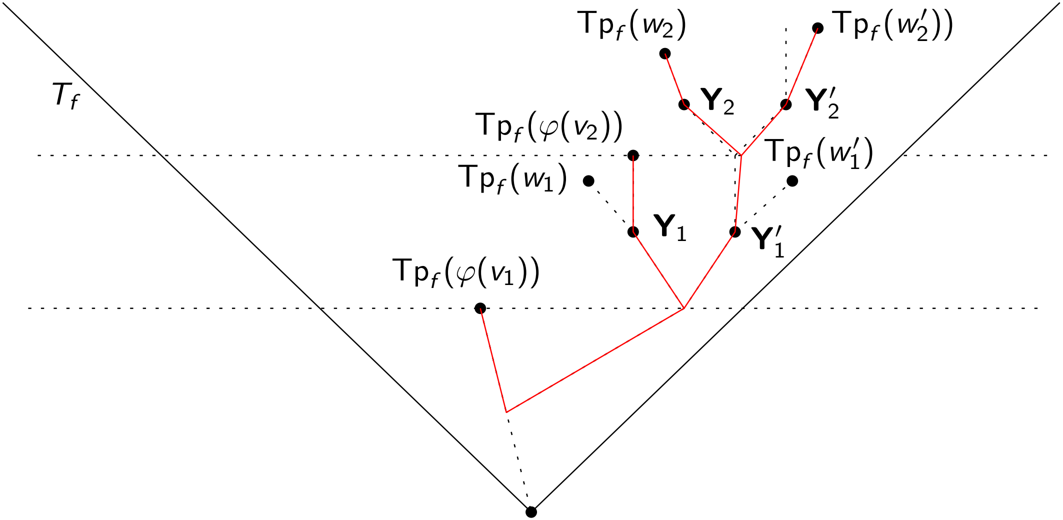
For every embedding $\varphi : \mathbf{R}^2_\omega \to \mathbf{R}^2_\omega$ *there exists an integer* m *such that for every* $n > m$ *there exists an edge* (v, w) *of label* 0 *in the image of* φ *of height* n.

Persistent colouring of edges



Thank you for the attention

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