Explicit construction of universal structures

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Joint work with Jarik Nešetřil

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Universal relational structures

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Let $\mathcal{C}$ be class of relational structures.

Definition

Relational structure $\mathbf{U}$ is *(embedding-)universal* for class $\mathcal{C}$ iff $\mathbf{U} \in \mathcal{C}$ and every structure $\mathbf{A} \in \mathcal{C}$ is induced substructure of $\mathbf{U}$.

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- **Universal graph**:
  - **Fraïssé**: homogeneous universal graph constructed by Fraïssé limit .
  - **Erdős and Rényi, 1963**: The countable random graph.
  - **Rado, 1965**: Explicit description:
    - **Vertices:** all finite 0–1 sequences $(a_1, a_2, \ldots, a_t), t \in \mathbb{N}$
    - **Edges:** $\{(a_1, a_2, \ldots, a_t), (b_1, b_2, \ldots, b_s)\}$ form edge

$$\iff$$

$b_a = 1$ where $a = \sum_{i=1}^{t} a_i 2^i$.

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$b_a = 1$ where $a = \sum_{i=1}^{t} a_i 2^i$.

  - Many variants of Rado's description are known.
  - All the description give up to isomorphism unique graph, as can be shown using the extension property.

Even more famous example

- **Class**: linear orders
- **Universal structure:** $\mathbb{Q}$.

Universal partial order

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- **Sketch of explicit description** (H., Nešetřil, 2003):

  Notation: Pairs $M = (M_L | M_R)$. M_L, M_R are sets.

  Vertices: Pair M is a vertex iff:

  1. (left completeness) $A_L \subseteq M_L$ for each $A \in M_L$,
  2. (right completeness) $B_R \subseteq M_R$ for each $B \in M_R$,
  3. (correctness)
     1. Elements M_L and M_R are vertices,
     2. $M_L \cap M_R = \emptyset$,
  4. (ordering property) $(\{A\} \cup A_R) \cap (\{B\} \cup B_L) \neq \emptyset$ for each $A \in M_L, B \in M_R$,

  Relation: We put $M < N$ if $(\{M\} \cup M_R) \cap (\{N\} \cup N_L) \neq \emptyset$.

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- Correspondence to Conway's surreal numbers.
- Later generalized to rational metric space (in H., Nešetřil, 2008; in constructive setting Lešnik, 2008).

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However there are positive examples of universal partial order.

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  no: $A = \{0\}, B = \{00, 01\}$ form a gap.

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- We prove by induction that there is winning strategy for Alice. Basic idea is “Venn diagram” property.

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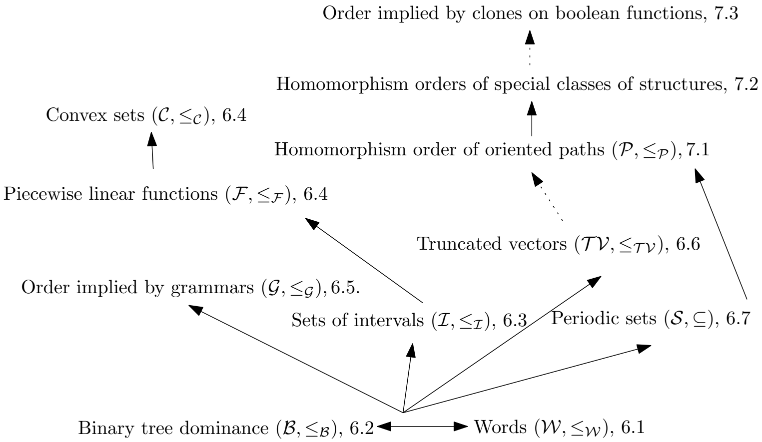
New proof (H., Nešetřil, 2011) by embedding periodic sets of natural numbers.

Catalogue of universal partial orders



Universal but not homogeneous

- **Hajnal, Pach, 1981**:
  - Nonexistence of universal 4-cycle-free graph.
- **Komjáth, Mekler, Pach, 1988**:
  - Existence of universal graph $\mathbf{P}_l$-free graph ($\mathbf{P}_l$ is graph of length l).
  - Existence of universal graph for classes without short odd cycles (fixed proof appears in 1999).
- **Covington, 1989**:
  - Existence of universal graph for class of graphs without induced path on 4 vertices.
  - Notion of amalgamation failure.
- **Komjáth, 1999**:
  - Existence of universal bowtie-free graph.
- **Cherlin, Shelah, Shi, 1999**:
  - Characterization of universal ω-categorical $\mathcal{F}$-free graphs via algebraic closure.
  - Existence of universal graph for classes defined by forbidden homomorphisms.
  - New examples

Universal graph without odd cycles of length at most $2l+1$

M-Structure is structure $\mathbf{M} = (V, G, F_1, \ldots, F_{2l+1})$ such that:

1. G is graph on V without loops;
2. $F_1, \ldots F_{2l+1}$ graph on V with loops;
3. $F_1 = G$;
4. $xy \in F_a, yz \in F_b, a+b \leq 2s+1$, then $xz \in F_{a+b}$;
5. if $a+b \leq 2s+1$ odd, then $F_a \cup G_b = \emptyset$.

Universal graph:

- Retract of Fraïssé limit of M structures.

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- Metric graph

Universal graph with forbidden induced path on 4 vertices

Covington's construction of a universal structure for class $\mathcal{C}$:

1. Identification of finite set of amalgamation failures
2. Extending language by new relations (homogenization), class $\mathcal{C}'$
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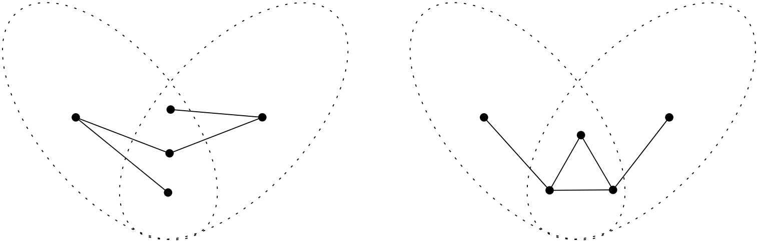

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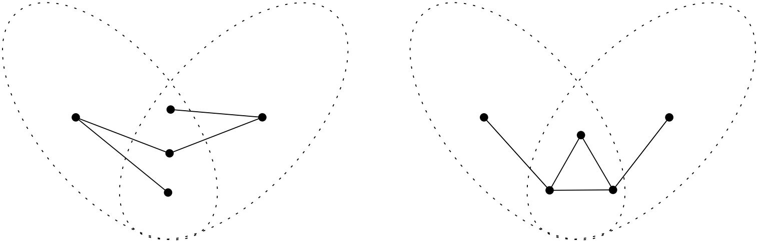

Language of graphs needs to be extended by single ternary relation.

Forbidden homomorphisms

$\mathcal{F}$ family of connected finite relational structures.

Class $Forb_h(\mathcal{F})$ consists of all relational structures $\mathbf{A}$ such that there is no homomorphism $\mathbf{F} \to \mathbf{A}, \mathbf{F} \in \mathcal{F}$.

Corollary (Cherlin, Shelah, Shi 1999)

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Cherlin, Shelah, Shi give an condition on existence of universal ω categorical structure for $\mathcal{F}$-free graphs.

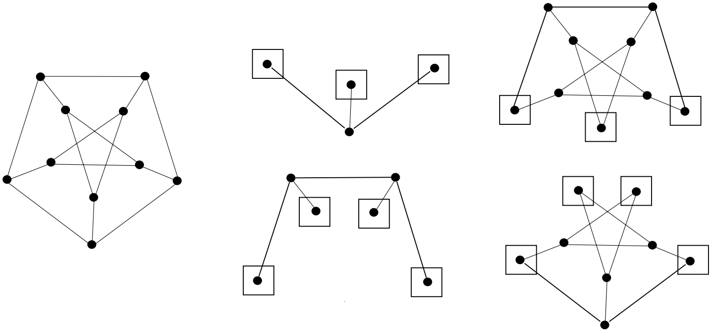
Explicit amalgamation argument for existence of universal graph for $Forb_h(F)$

Definition

For relational structure **A** and inclusion minimal vertex cut C in its Gaifman graph of **A**, a *piece of relational structure* **A** is pair $\mathcal{P} = (\mathbf{P}, \overrightarrow{C})$.
Here **P** is structure induced on **A** by union of C and vertices of some connected component of $\mathbf{A} \setminus C$.
Tuple $\overrightarrow{C}$ consist of the vertices of cut C in (arbitrary) linear order.
Vertices C are *roots of piece* $\mathcal{P}$.

Examples

- The pieces of Petersen graph



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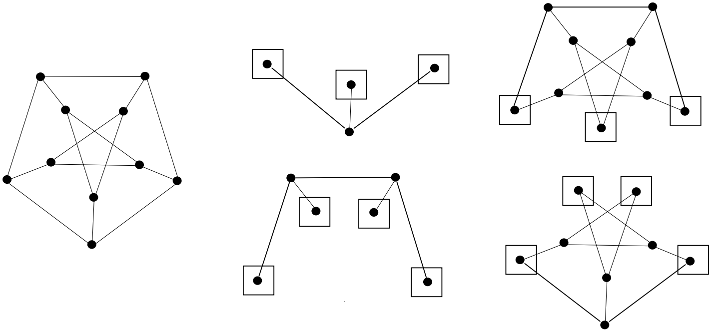
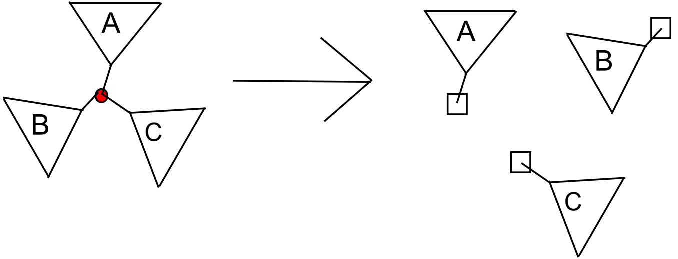


- Pieces of cycles of length n = paths of length $2, \ldots, n-2$ rooted at both ends.

- Pieces of a relational tree **T** = branches of **T**.



Proof (sketch)

- Enumerate all pieces of all forbidden structures $\mathbf{F} \in \mathcal{F}$ as $\mathcal{P}_1 = (\mathbf{P}_1, \overrightarrow{C}_1), \ldots, \mathcal{P}_N = (\mathbf{P}_N, \overrightarrow{C}_N)$.

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  - Existence of homomorphism $f : \mathbf{P_i} \to \mathbf{R}$ imply $f(\overrightarrow{C}_i) \in X_i$.
  - Let $\mathcal{P}_{i_1}, \ldots \mathcal{P}_{i_n}$ be all pieces generated by cut $\overrightarrow{C}$.
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- Substructures of expansions of all $\mathbf{R} \in \mathit{Forb}_h(\mathcal{F})$ form an amalgamation class.
  Reduct of the Fraïssé limit of this class is an universal graph for $\mathit{Forb}_h(\mathcal{F})$.

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    - Universal graph is “blown up” finite graph: explicit description is easy.

Study of specific examples

The arity of new relation depend on the size of inclusion minimal vertex cuts of Gaifman graph.

The construction is optimal: expansions with smaller arities will not give amalgamation class.

- Arity 1 (monadic expansion)
  - Expansion equivalent to a vertex coloring
  - (Relational) trees:
    - Expansion is axiomatized by forbidden edges.
    - Universal graph retracted by unifying vertices of the same color is homomorphism-universal graph.
    - Can be seen as a new construction of homomorphism duals.
    - Universal graph is “blown up” finite graph: explicit description is easy.
  - Relations **F** such that their Gaifman graph is simple:
    - Forbidden irreducible structures.
    - No finite homomorphism universal object
    - Blown up finite graph, with forbidden cliques.

Study of specific examples

- Arity 2: forbidden cycles, etc.
  - Representation translate to a metric space
  - Explicit construction of metric space can be directly used to represent these.
- Beyond arity 2 explicit representation still possible, but impractical to describe in full generality.

Thank you. . .

. . . Questions?