All those Ramsey classes

Ramsey classes with closures and forbidden homomorphisms

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Joint work with Jaroslav Nešetřil

Logic Colloquium 2016, Leeds

Ramsey Theorem

Theorem (Ramsey Theorem, 1930)

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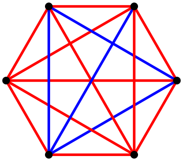

For $p = 2$, $n = 3$, $k = 2$ put $N = 6$

Many aspects of Ramsey theorem

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$\swarrow$

Logic

Many aspects of Ramsey theorem

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Ramsey theorem for finite relational structures

Let L be a purely relational language with binary relation $\leq$.

Denote by $\overrightarrow{Rel}(L)$ the class of all finite L-structures where $\leq$ is a linear order.

Theorem (Nešetřil-Rödl, 1977; Abramson-Harrington, 1978)

$$\forall_{\mathbf{A},\mathbf{B}\in\overrightarrow{Rel}(L)}\exists_{\mathbf{C}\in\overrightarrow{Rel}(L)} : \mathbf{C} \longrightarrow (\mathbf{B})^{\mathbf{A}}_2.$$

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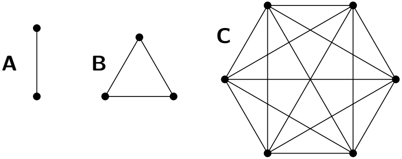

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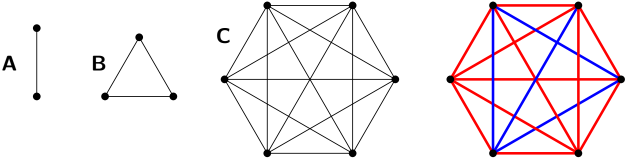

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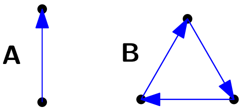

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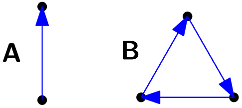
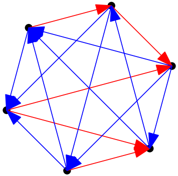


Vertices of **C** can be linearly ordered and edges coloured accordingly:

- If edge is goes forward in linear order it is red
- blue otherwise.



Structural extensions

Ramsey theorem for finite relational structures

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Structures
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Structural extensions

Ramsey theorem for finite relational structures

Structures with functions (Models)

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Structural extensions

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Structural extensions

Ramsey theorem for finite relational structures

- Structures with functions (Models)
- Structures with axioms
- Infinite structures
- Categories

Ramsey theorem for finite models

Let L be a language with both relations and functions.

Assume that L contains binary relation $\leq$.

Denote by $\overrightarrow{Mod}(L)$ the class of all finite L-structures where $\leq$ is a linear order.

Theorem (H.-Nešetřil, 2016)

$$\forall_{\mathbf{A},\mathbf{B}\in\overrightarrow{Mod}(L)}\exists_{\mathbf{C}\in\overrightarrow{Mod}(L)} : \mathbf{C} \longrightarrow (\mathbf{B})^{\mathbf{A}}_2.$$

Ramsey classes

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A class $\mathcal{C}$ of finite L-structures is Ramsey iff $\forall_{\mathbf{A},\mathbf{B}\in\mathcal{C}}\exists_{\mathbf{C}\in\mathcal{C}} : \mathbf{C} \longrightarrow (\mathbf{B})^{\mathbf{A}}_2$.

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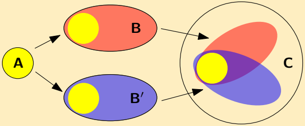
Ramsey classes are amalgamation classes

Definition (Amalgamation property of class $\mathcal{K}$)



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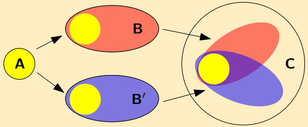
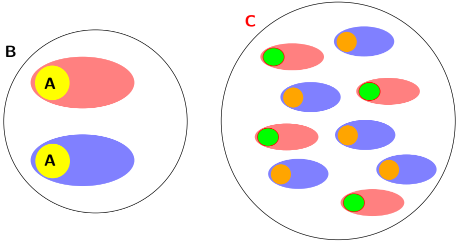


Nešetřil, 80's: Under mild assumptions Ramsey classes have amalgamation property.



Nešetřil's Classification Programme, 2005

Classification Programme

Ramsey classes	$\Longrightarrow$	amalgamation classes
$\Uparrow$		$\Downarrow$
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Theorem (Jasiński, Laflamme, Nguyen Van Thé, Woodrow, 2014)

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Let $\mathcal{K}$ be class of L-structures and $\mathcal{K}'$ be class of lifts of $\mathcal{K}$.

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Theorem (Kechris, Pestov, Todorčevič 2005, Nguyen van Thé 2012)

For every amalgamation class $\mathcal{K}$ there exists, up to bi-definability, at most one Ramsey class $\mathcal{K}'$ of lifts of $\mathcal{K}$ with lift property.

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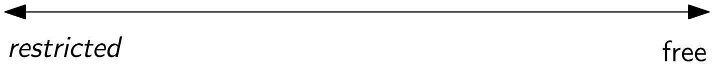
Theorem (Evans, Hubička, Nešetřil, 2016+)

There still exists minimal (non-precompact) Ramsey lift of M with lift property.

Map of Ramsey Classes



Map of Ramsey Classes

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Map of Ramsey Classes

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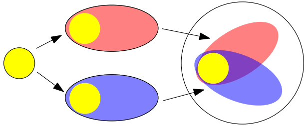
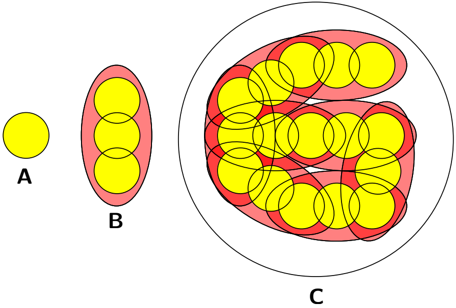
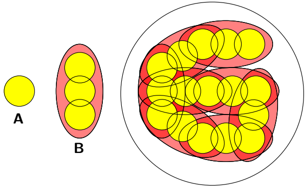
From an amalgamation class to a Ramsey class



The Nešetřil-Rödl partite construction of Ramsey object demands more complicated (multi)amalgamations.



$\langle \mathbf{A}, \mathbf{B}, \mathbf{C} \rangle$-hypergraphs



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Given structures **A**, **B**, **C** the $\langle \mathbf{A}, \mathbf{B}, \mathbf{C} \rangle$-hypergraph is a hypergraph whose vertices are copies of **A** in **C** and hyper-edges are corresponds to copies of **B**: M is an hyper-edge if $M = \widetilde{\mathbf{B}}$ for some $\widetilde{\mathbf{B}} \in \binom{\mathbf{C}}{\mathbf{B}}$

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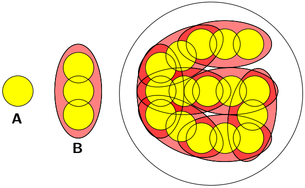

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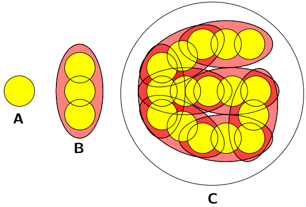

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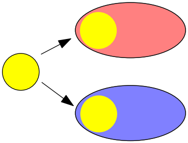
Hypergraphs of large chromatic number are known to exists, but typically they do not correspond to $\langle \mathbf{A}, \mathbf{B}, \mathbf{C} \rangle$-hypergraph

Our approach



Structure is irreducible if every pair of vertices is contained in some tuple of some relation

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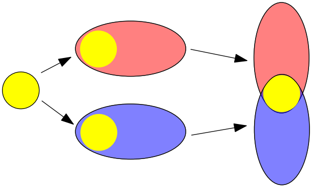

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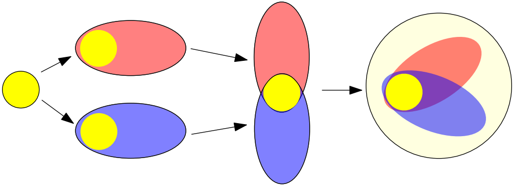

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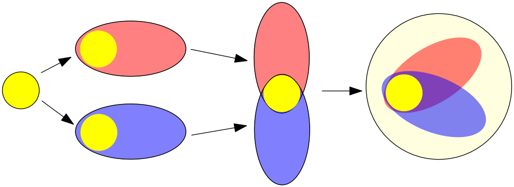

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Definition

Irreducible structure $\mathbf{C}'$ is a strong completion of $\mathbf{C}$ if it has the same vertex set and every irreducible substructure of $\mathbf{C}$ is also (induced) substructure of $\mathbf{C}'$.

Our approach

Theorem (H.-Nešetřil, 2016)

Let $\mathcal{R}$ be a Ramsey class of irreducible finite structures and let $\mathcal{K}$ be a hereditary locally finite subclass of $\mathcal{R}$ with strong amalgamation. Then $\mathcal{K}$ is Ramsey.

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Schematically

$$\text{Ramsey} \implies \text{amalgamation}$$
$$\text{amalgamation + order + local finiteness} \implies \text{Ramsey}$$

What is local finiteness?

Our approach

$\mathcal{K}$ is locally finite subclass of $\mathcal{R}$ if for every $\mathbf{C}_0$ in $\mathcal{R}$ there exists a finite bound on minimal set of obstacles which prevents a structure with homomorphism to $\mathbf{C}_0$ from being completed to $\mathcal{K}$.

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Definition

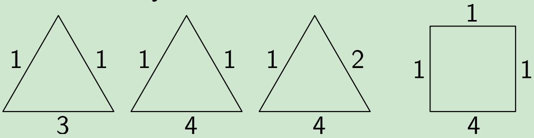
Let $\mathcal{R}$ be a class of finite irreducible structures and $\mathcal{K}$ a subclass of $\mathcal{R}$. We say that the class $\mathcal{K}$ is locally finite subclass of $\mathcal{R}$ if for every $\mathbf{C}_0 \in \mathcal{R}$ there is $n = n(\mathbf{C}_0)$ such that every structure $\mathbf{C}$ has strong $\mathcal{K}$-completion providing that it satisfies the following:

1. there is a homomorphism-embedding from $\mathbf{C}$ to $\mathbf{C}_0$
2. every substructure of $\mathbf{C}$ with at most n vertices has a strong $\mathcal{K}$-completion.

Locally finite subclass, an example

Example

Consider class of metric spaces with distances $\{1, 2, 3, 4\}$. Graph with edges labelled by $\{1, 2, 3, 4\}$ can be completed to a metric space if and only if it does not contain one of:



S-metric-spaces

Theorem (H.-Nešetřil, 2016)

Let S be set of positive reals. The class $\mathcal{M}_S$ of all metric spaces with distances restricted to S has precompact Ramsey lift iff $\mathcal{M}_S$ is an amalgamation class.

- Consider only finite S
- Identify minimal obstacles for completion of S-graphs.
- Show that they are all cycles of bounded size
- Apply previous theorem.

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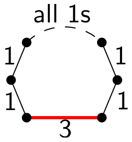
Consider $S = \{1, 3\}$ and cycles:



S-metric-spaces

Definition (Delhommé, Laflamme, Pouzet, Sauer, 2007)

$a \in S$ is jump number if $a < \max(S)$ and $2a \leq \min_{b \in S}(b > a)$.

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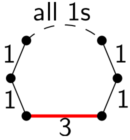
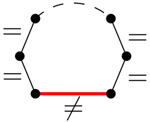
Lemma

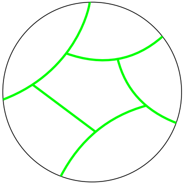
If $\mathcal{K}$ is a subclass of $\mathcal{R}$ and define more equivalences then $\mathcal{R}$, then $\mathcal{K}$ is not locally finite subclass.





Extended approach

- Represent equivalences with infinitely many classes by additional vertices and functions



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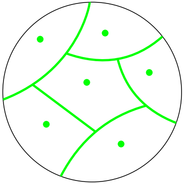

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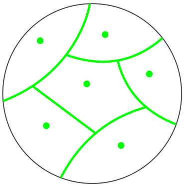
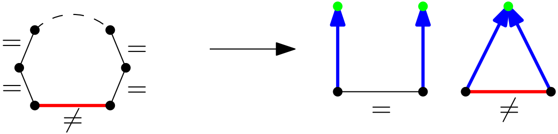


- Show local finiteness in this extended language. Forbidden configurations are now finite:



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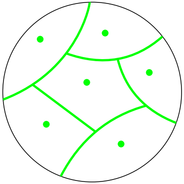

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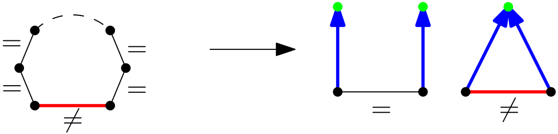

- We need Ramsey theorem for amalgamation classes of ordered models!

Ramsey theorem with closures

Definition

Let L be a language, $\mathcal{R}$ be a Ramsey class of finite irreducible L-structures and $\mathcal{U}$ be a closure description. We say that a subclass $\mathcal{K}$ of $\mathcal{R}$ is an $(\mathcal{R},\mathcal{U})$-multiamalgamation class iff:

1. $\mathcal{K}$ is hereditary subclass of $\mathcal{R}$ consiting of $\mathcal{U}$-closed structures.
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1. $\mathcal{K}$ is hereditary subclass of $\mathcal{R}$ consiting of $\mathcal{U}$-closed structures.
2. $\mathcal{K}$ is closed for strong amalgamation over $\mathcal{U}$-closed substructures
3. Let $\mathbf{B} \in \mathcal{K}$ and $\mathbf{C}_0 \in \mathcal{R}$. Then there exists $n = n(\mathbf{B}, \mathbf{C}_0)$ such that if $\mathcal{U}$-closed L-structure $\mathbf{C}$ satisfies the following:
    1. $\mathbf{C}_0$ is a completion of $\mathbf{C}$), and,
    2. every substructure of $\mathbf{C}$, $|C| \leq n$ has a $\mathcal{K}$-completion.

    Then there exists $\mathbf{C}' \in \mathcal{K}$ that is a completion of $\mathbf{C}$ with respect to copies of $\mathbf{B}$.

Ramsey theorem with closures

Theorem (H.-Nešetřil, 2016)

Every $(\mathcal{R},\mathcal{U})$-multiamalgamation class $\mathcal{K}$ is Ramsey.

Again it is the only non-trivial step is to verify local finiteness!

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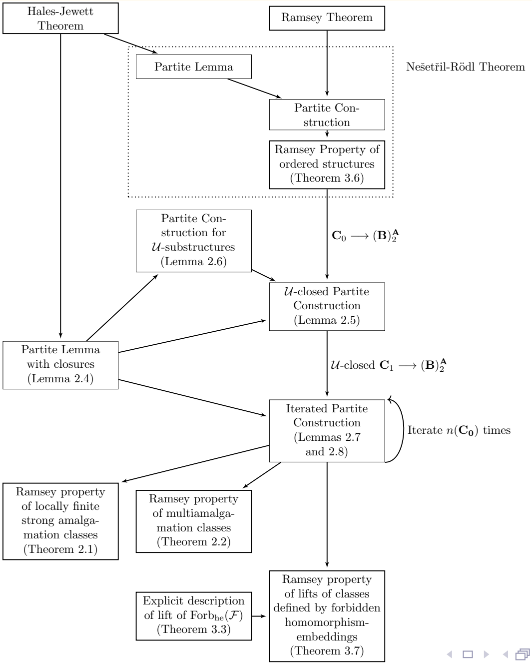
The theorem is proved by a variation and strengthening of the Partite Constructions (Nešetřil-Rödl).


Hales-Jewett Theorem
Ramsey Theorem
Partite Lemma
Nešetřil-Rödl Theorem
Partite Construction
Ramsey Property of ordered structures (Theorem 3.6)
Partite Construction for $\mathcal{U}$-substructures (Lemma 2.6)
$\mathbf{C}_0 \longrightarrow (\mathbf{B})^{\mathbf{A}}_2$
$\mathcal{U}$-closed Partite Construction (Lemma 2.5)
Partite Lemma with closures (Lemma 2.4)
$\mathcal{U}$-closed $\mathbf{C}_1 \longrightarrow (\mathbf{B})^{\mathbf{A}}_2$
Iterated Partite Construction (Lemmas 2.7 and 2.8)
Iterate $n(\mathbf{C_0})$ times
Ramsey property of locally finite strong amalgamation classes (Theorem 2.1)
Ramsey property of multiamalgamation classes (Theorem 2.2)
Explicit description of lift of $\mathrm{Forb}_{\mathrm{he}}(\mathcal{F})$ (Theorem 3.3)
Ramsey property of lifts of classes defined by forbidden homomorphism-embeddings (Theorem 3.7)

Ramsey lifts for forbidden homomorphisms

Theorem (Nešetřil-Rödl Theorem for relational structures, 1977)

Let L be a relational language containing binary relation $\leq$ and $\mathcal{E}$ be a (possibly infinite) family of ordered irreducible L-structures.

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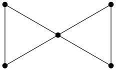
Then the class $\mathcal{K}$ of all ordered structures in $\mathrm{Forb_{he}}(\mathcal{F})$ *has a precompact Ramsey lift.*

Example

Theorem (H., Nešetřil, 2014)

The class of graphs not containing bow-tie as non-induced subgraph have Ramsey lift.

Bow-tie graph:



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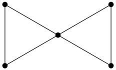
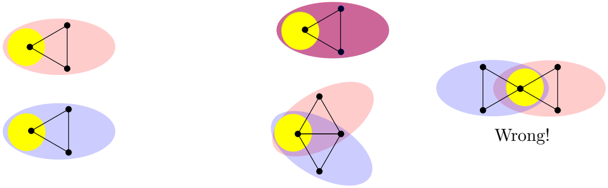
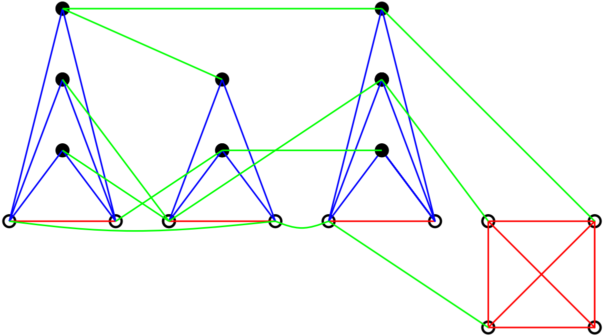


Amalgamation of two triangles must unify vertices.



Example

Structure of bow-tie-free graphs



Edges in no triangles Edges in 1 triangle Edges in >1 triangles

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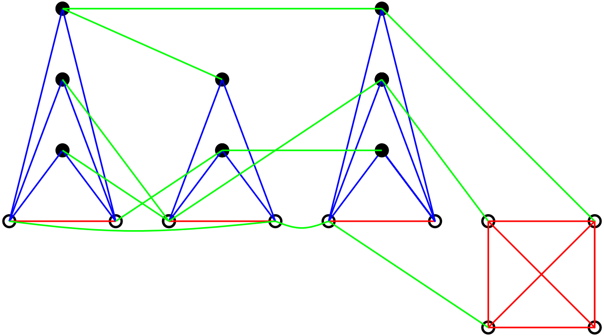

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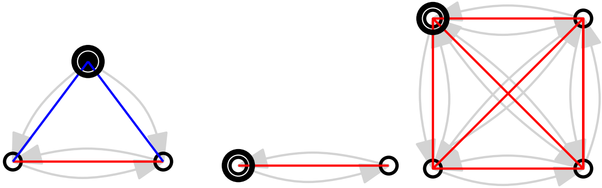
Lemma

Bow-tie-free graphs have free amalgamation over closed structures.

Example

Closures in the class of bowtie-free graphs:



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Theorem (H.-Nešetřil, 2016)

Let $\mathcal{M}$ be a set of finite connected structures such that $\mathrm{Forb_m}(\mathcal{M})$ *has an ω-categorical universal structure* $\mathbf{U}$.

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Then the class of finite algebraically closed substructures of $\mathbf{U}$ has precompact Ramsey lift.

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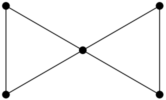

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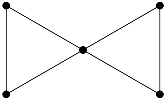

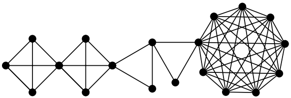

Further applications

- Many classical Ramsey classes follows as corollaries of our results:
  - Partial orders
  - Metric spaces
  - H-colourable graphs
  - Graphs with metric embeddings
  - Steiner systems
  - graphs with no short odd cycles
  - …
- S-metric spaces and generalizations
- Classes defining multiple orders
- Totally ordered structures
- “Fat” structures
- …

Thank you for the attention

- J.H., J. Nešetřil: Bowtie-free graphs have a Ramsey lift. Submitted (arXiv:1402.2700), 2014, 22 pages.
- J.H., J. Nešetřil: All those Ramsey classes (Ramsey classes with closures and forbidden homomorphisms). Submitted (arXiv:1606.07979), 2016, 59 pages.