All those Ramsey classes

Ramsey classes with closures and forbidden homomorphisms

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Joint work with Jaroslav Nešetřil

Logic Colloquium 2016, Leeds

Ramsey Theorem

Theorem (Ramsey Theorem, 1930)

$$\forall_{n,p,k\geq 1}\exists_N : N \longrightarrow (n)^p_k.$$

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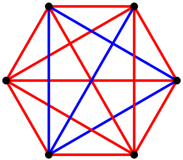

For $p = 2$, $n = 3$, $k = 2$ put $N = 6$

Many aspects of Ramsey theorem

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Logic

Many aspects of Ramsey theorem

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Combinatorics

Many aspects of Ramsey theorem

Ramsey theorem

Logic Combinatorics Model Theory

Many aspects of Ramsey theorem

$$
\begin{tikzcd}
 & & \text{Ramsey theorem} \arrow[dll] \arrow[dl] \arrow[d] \arrow[dr] & \\
\text{Logic} & \text{Combinatorics} & \text{Model Theory} & \text{Topological dynamics}
\end{tikzcd}
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Ramsey theorem for finite relational structures

Let L be a purely relational language with binary relation $\leq$.

Denote by $\overrightarrow{Rel}(L)$ the class of all finite L-structures where $\leq$ is a linear order.

Theorem (Nešetřil-Rödl, 1977; Abramson-Harrington, 1978)

$$\forall_{\mathbf{A},\mathbf{B}\in\overrightarrow{Rel}(L)} \exists_{\mathbf{C}\in\overrightarrow{Rel}(L)} : \mathbf{C} \longrightarrow (\mathbf{B})^{\mathbf{A}}_2.$$

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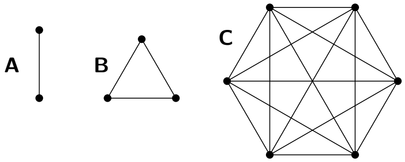

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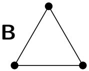
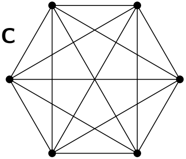
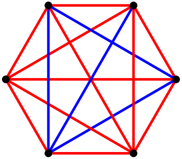
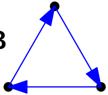
A  **B**  **C**  

Order is necessary

A

B


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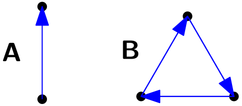
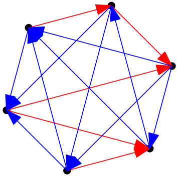


Vertices of **C** can be linearly ordered and edges coloured accordingly:

- If edge is goes forward in linear order it is red
- blue otherwise.



Structural extensions

Ramsey theorem for finite relational structures

Structural extensions

Ramsey theorem for finite relational structures

Structures
with
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(Models)

Structural extensions

Ramsey theorem for finite relational structures

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Infinite structures

Structural extensions

Ramsey theorem for finite relational structures

- Structures with functions (Models)
- Structures with axioms
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- Categories

Ramsey theorem for finite models

Let L be a language with both relations and functions.

Assume that L contains binary relation $\leq$.

Denote by $\overrightarrow{Mod}(L)$ the class of all finite L-structures where $\leq$ is a linear order.

Theorem (H.-Nešetřil, 2016)

$$\forall_{\mathbf{A},\mathbf{B}\in\overrightarrow{Mod}(L)}\exists_{\mathbf{C}\in\overrightarrow{Mod}(L)} : \mathbf{C} \longrightarrow (\mathbf{B})^{\mathbf{A}}_2.$$

Ramsey classes

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A class $\mathcal{C}$ of finite L-structures is Ramsey iff $\forall_{\mathbf{A},\mathbf{B}\in\mathcal{C}}\exists_{\mathbf{C}\in\mathcal{C}} : \mathbf{C} \longrightarrow (\mathbf{B})^{\mathbf{A}}_2$.

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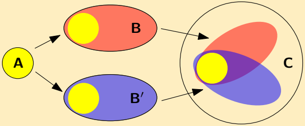
Ramsey classes are amalgamation classes

Definition (Amalgamation property of class $\mathcal{K}$)



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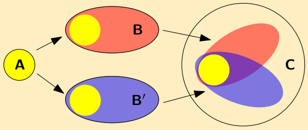
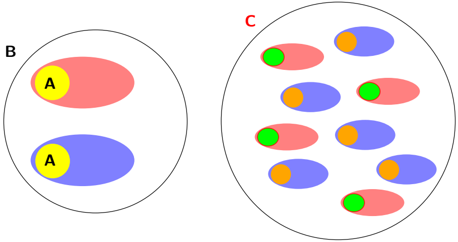


Nešetřil, 80’s: Under mild assumptions Ramsey classes have amalgamation property.



Nešetřil's Classification Programme, 2005

Classification Programme

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\text{Ramsey classes} & \Longrightarrow & \text{amalgamation classes} \\
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Theorem (Jasiński, Laflamme, Nguyen Van Thé, Woodrow, 2014)

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Does every amalgamation class have a Ramsey lift?

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Let $\mathcal{K}$ be class of L-structures and $\mathcal{K}'$ be class of lifts of $\mathcal{K}$.

- $\mathcal{K}$ is precompact if for every $\mathbf{A} \in \mathcal{K}$ there are only finitely many lifts of $\mathbf{A}$ in $\mathcal{K}'$.

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Theorem (Kechris, Pestov, Todorčevič 2005, Nguyen van Thé 2012)

For every amalgamation class $\mathcal{K}$ there exists, up to bi-definability, at most one Ramsey class $\mathcal{K}'$ of lifts of $\mathcal{K}$ with lift property.

Existence of precompact lifts

Question: Does every amalgamation class have a precompact Ramsey lift?

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Theorem (Evans, Hubička, Nešetřil, 2016+)

There still exists minimal (non-precompact) Ramsey lift of M with lift property.

Map of Ramsey Classes

restricted $\longleftrightarrow$ free

Map of Ramsey Classes

interval graphs

permutations

cyclic orders

unions of complete graphs

linear orders

restricted free

Map of Ramsey Classes

interval graphs

metric spaces

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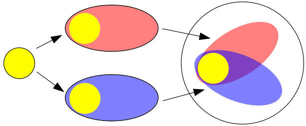
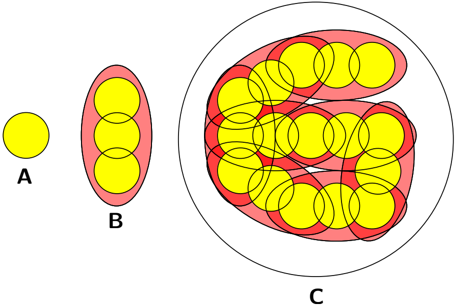
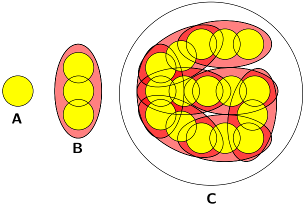
From an amalgamation class to a Ramsey class



The Nešetřil-Rödl partite construction of Ramsey object demands more complicated (multi)amalgamations.



$\langle \mathbf{A}, \mathbf{B}, \mathbf{C} \rangle$-hypergraphs



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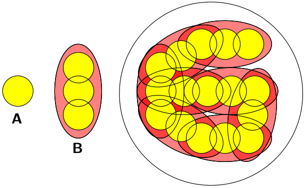

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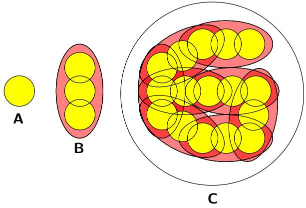

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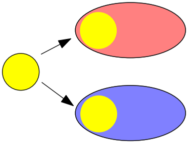
If the $\langle \mathbf{A}, \mathbf{B}, \mathbf{C} \rangle$-hypegraph has chromatic number 3, then $C \longrightarrow (\mathbf{B})^{\mathbf{A}}_2$

Hypergraphs of large chromatic number are known to exists, but typically they do not correspond to $\langle \mathbf{A}, \mathbf{B}, \mathbf{C} \rangle$-hypergraph

Our approach



Structure is irreducible if every pair of vertices is contained in some tuple of some relation

We consider classes of irreducible structures and split amalgamation into two steps:

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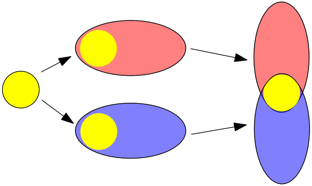

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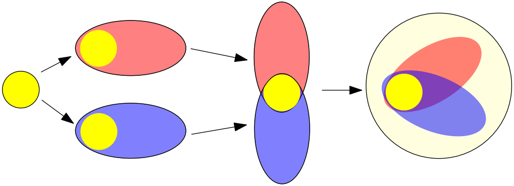

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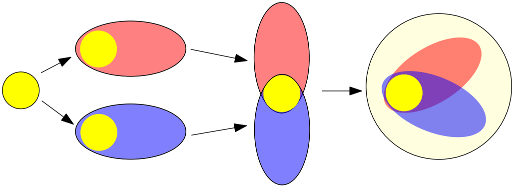

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Definition

Irreducible structure $\mathbf{C}'$ is a strong completion of $\mathbf{C}$ if it has the same vertex set and every irreducible substructure of $\mathbf{C}$ is also (induced) substructure of $\mathbf{C}'$.

Our approach

Theorem (H.-Nešetřil, 2016)

Let $\mathcal{R}$ be a Ramsey class of irreducible finite structures and let $\mathcal{K}$ be a hereditary locally finite subclass of $\mathcal{R}$ with strong amalgamation. Then $\mathcal{K}$ is Ramsey.

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Schematically

$$\text{Ramsey} \implies \text{amalgamation}$$
$$\text{amalgamation + order + local finiteness} \implies \text{Ramsey}$$

What is local finiteness?

Our approach

$\mathcal{K}$ is locally finite subclass of $\mathcal{R}$ if for every $\mathbf{C}_0$ in $\mathcal{R}$ there exists a finite bound on minimal set of obstacles which prevents a structure with homomorphism to $\mathbf{C}_0$ from being completed to $\mathcal{K}$.

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Definition

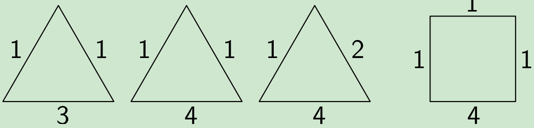
Let $\mathcal{R}$ be a class of finite irreducible structures and $\mathcal{K}$ a subclass of $\mathcal{R}$. We say that the class $\mathcal{K}$ is locally finite subclass of $\mathcal{R}$ if for every $\mathbf{C}_0 \in \mathcal{R}$ there is $n = n(\mathbf{C}_0)$ such that every structure $\mathbf{C}$ has strong $\mathcal{K}$-completion providing that it satisfies the following:

1. there is a homomorphism-embedding from $\mathbf{C}$ to $\mathbf{C}_0$
2. every substructure of $\mathbf{C}$ with at most n vertices has a strong $\mathcal{K}$-completion.

Locally finite subclass, an example

Example

Consider class of metric spaces with distances $\{1, 2, 3, 4\}$.
Graph with edges labelled by $\{1, 2, 3, 4\}$ can be completed to a metric space if and only if it does not contain one of:



S-metric-spaces

Theorem (H.-Nešetřil, 2016)

Let S be set of positive reals. The class $\mathcal{M}_S$ of all metric spaces with distances restricted to S has precompact Ramsey lift iff $\mathcal{M}_S$ is an amalgamation class.

- Consider only finite S
- Identify minimal obstacles for completion of S-graphs.
- Show that they are all cycles of bounded size
- Apply previous theorem.

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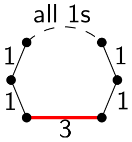
Consider $S = \{1, 3\}$ and cycles:



S-metric-spaces

Definition (Delhommé, Laflamme, Pouzet, Sauer, 2007)

$a \in S$ is jump number if $a < \max(S)$ and $2a \leq \min_{b \in S}(b > a)$.

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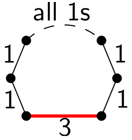
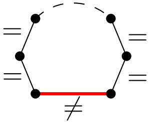
Lemma

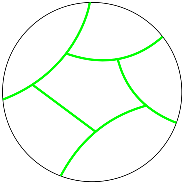
If $\mathcal{K}$ is a subclass of $\mathcal{R}$ and define more equivalences then $\mathcal{R}$, then $\mathcal{K}$ is not locally finite subclass.





Extended approach

- Represent equivalences with infinitely many classes by additional vertices and functions



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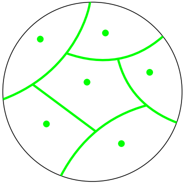

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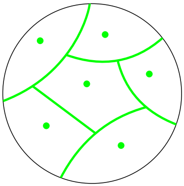
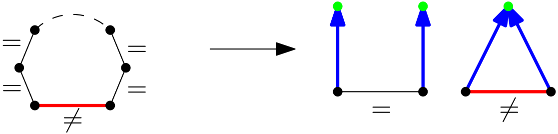


- Show local finiteness in this extended language. Forbidden configurations are now finite:



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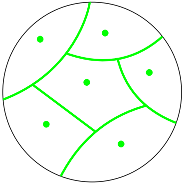

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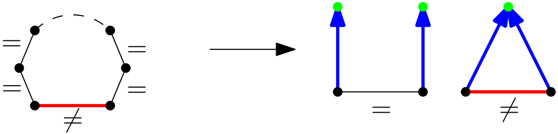

- We need Ramsey theorem for amalgamation classes of ordered models!

Ramsey theorem with closures

Definition

Let L be a language, $\mathcal{R}$ be a Ramsey class of finite irreducible L-structures and $\mathcal{U}$ be a closure description. We say that a subclass $\mathcal{K}$ of $\mathcal{R}$ is an $(\mathcal{R}, \mathcal{U})$-multiamalgamation class iff:

1. $\mathcal{K}$ is hereditary subclass of $\mathcal{R}$ consiting of $\mathcal{U}$-closed structures.
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1. $\mathcal{K}$ is hereditary subclass of $\mathcal{R}$ consiting of $\mathcal{U}$-closed structures.
2. $\mathcal{K}$ is closed for strong amalgamation over $\mathcal{U}$-closed substructures
3. Let $\mathbf{B} \in \mathcal{K}$ and $\mathbf{C}_0 \in \mathcal{R}$. Then there exists $n = n(\mathbf{B}, \mathbf{C}_0)$ such that if $\mathcal{U}$-closed L-structure $\mathbf{C}$ satisfies the following:
   1. $\mathbf{C}_0$ is a completion of $\mathbf{C}$), and,
   2. every substructure of $\mathbf{C}$, $|C| \leq n$ has a $\mathcal{K}$-completion.

   Then there exists $\mathbf{C}' \in \mathcal{K}$ that is a completion of $\mathbf{C}$ with respect to copies of $\mathbf{B}$.

Ramsey theorem with closures

Theorem (H.-Nešetřil, 2016)

Every $(\mathcal{R}, \mathcal{U})$-multiamalgamation class $\mathcal{K}$ is Ramsey.

Again it is the only non-trivial step is to verify local finiteness!

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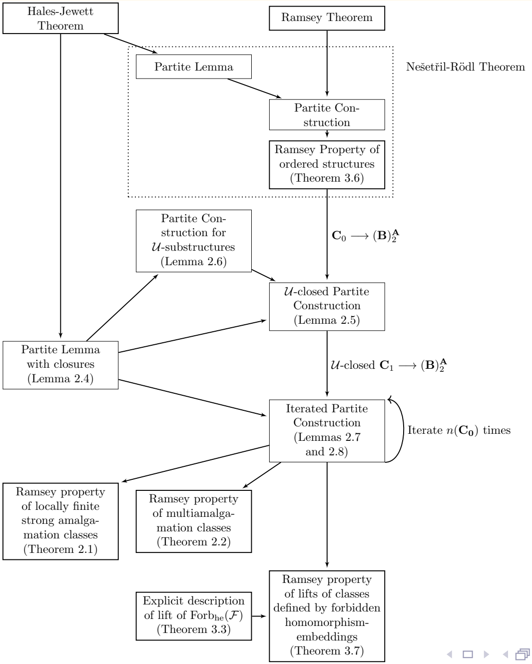
The theorem is proved by a variation and strengthening of the Partite Constructions (Nešetřil-Rödl).


Hales-Jewett Theorem
Ramsey Theorem
Partite Lemma
Nešetřil-Rödl Theorem
Partite Construction
Ramsey Property of ordered structures (Theorem 3.6)
Partite Construction for $\mathcal{U}$-substructures (Lemma 2.6)
$\mathbf{C}_0 \longrightarrow (\mathbf{B})^{\mathbf{A}}_2$
$\mathcal{U}$-closed Partite Construction (Lemma 2.5)
Partite Lemma with closures (Lemma 2.4)
$\mathcal{U}$-closed $\mathbf{C}_1 \longrightarrow (\mathbf{B})^{\mathbf{A}}_2$
Iterated Partite Construction (Lemmas 2.7 and 2.8)
Iterate $n(\mathbf{C_0})$ times
Ramsey property of locally finite strong amalgamation classes (Theorem 2.1)
Ramsey property of multiamalgamation classes (Theorem 2.2)
Explicit description of lift of $\mathrm{Forb}_{\mathrm{he}}(\mathcal{F})$ (Theorem 3.3)
Ramsey property of lifts of classes defined by forbidden homomorphism-embeddings (Theorem 3.7)

Ramsey lifts for forbidden homomorphisms

Theorem (Nešetřil-Rödl Theorem for relational structures, 1977)

Let L be a relational language containing binary relation $\leq$ and $\mathcal{E}$ be a (possibly infinite) family of ordered irreducible L-structures.

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Let L be a relational language containing binary relation $\leq$, and $\mathcal{F}$ be a regular family of finite connected weakly ordered L-structures.

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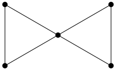
Then the class $\mathcal{K}$ of all ordered structures in $\mathrm{Forb_{he}}(\mathcal{F})$ *has a precompact Ramsey lift.*

Example

Theorem (H., Nešetřil, 2014)

The class of graphs not containing bow-tie as non-induced subgraph have Ramsey lift.

Bow-tie graph:



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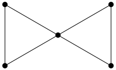
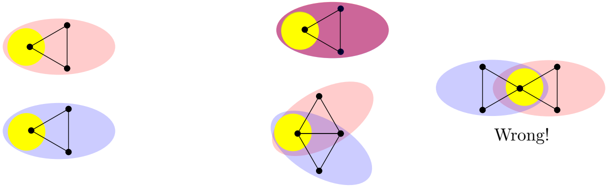
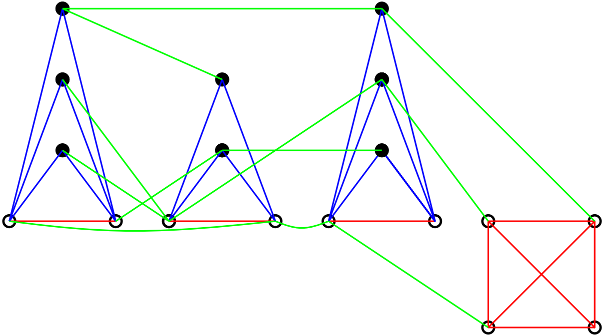


Amalgamation of two triangles must unify vertices.



Example

Structure of bow-tie-free graphs



Edges in no triangles Edges in 1 triangle Edges in $>$ 1 triangles

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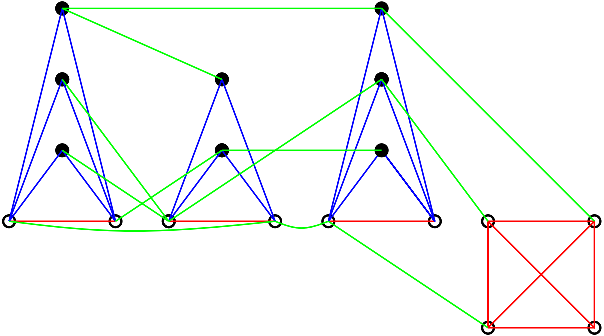


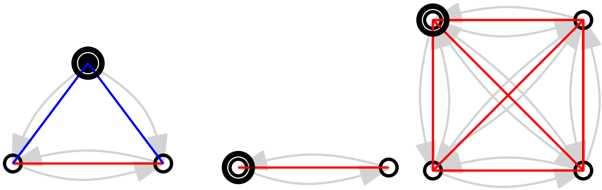
Edges in no triangles   Edges in 1 triangle   Edges in $>$ 1 triangles

Lemma

Bow-tie-free graphs have free amalgamation over closed structures.

Example

Closures in the class of bowtie-free graphs:



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Theorem (H.-Nešetřil, 2016)

Let $\mathcal{M}$ be a set of finite connected structures such that $\mathrm{Forb_m}(\mathcal{M})$ *has an* ω*-categorical universal structure* $\mathbf{U}$.

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Then the class of finite algebraically closed substructures of $\mathbf{U}$ has precompact Ramsey lift.

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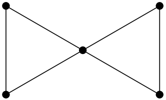

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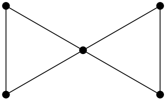

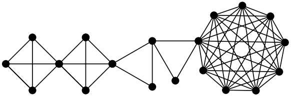

Further applications

- Many classical Ramsey classes follows as corollaries of our results:
  - Partial orders
  - Metric spaces
  - H-colourable graphs
  - Graphs with metric embeddings
  - Steiner systems
  - graphs with no short odd cycles
  - ...
- S-metric spaces and generalizations
- Classes defining multiple orders
- Totally ordered structures
- “Fat” structures
- ...

Thank you for the attention

- J.H., J. Nešetřil: Bowtie-free graphs have a Ramsey lift. Submitted (arXiv:1402.2700), 2014, 22 pages.
- J.H., J. Nešetřil: All those Ramsey classes (Ramsey classes with closures and forbidden homomorphisms). Submitted (arXiv:1606.07979), 2016, 59 pages.