Ramsey properties of Hrushovski construction

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Joint work with David Evans and Jaroslav Nešetřil

British Postgraduate Model Theory Conference, Leeds

Ramsey Theorem

Theorem (Ramsey Theorem, 1930)

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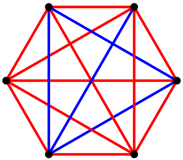

For $p = 2$, $n = 3$, $k = 2$ put $N = 6$

Many aspects of Ramsey theorem

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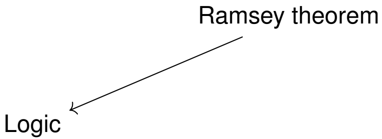

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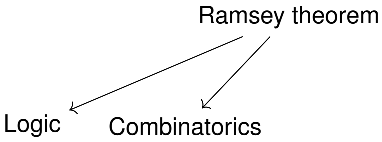

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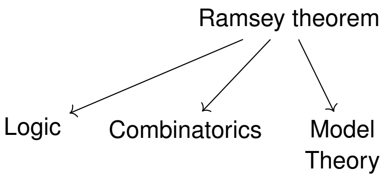

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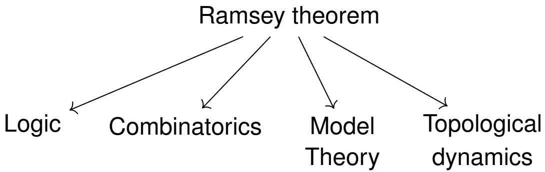

Ramsey theorem for finite relational structures

Let L be a purely relational language with binary relation $\leq$.

Denote by $\overrightarrow{Rel}(L)$ the class of all finite L-structures where $\leq$ is a linear order.

Theorem (Nešetřil-Rödl, 1977; Abramson-Harrington, 1978)

$$\forall_{\mathbf{A},\mathbf{B}\in\overrightarrow{Rel}(L)}\exists_{\mathbf{C}\in\overrightarrow{Rel}(L)}:\mathbf{C}\longrightarrow(\mathbf{B})^{\mathbf{A}}_{2}.$$

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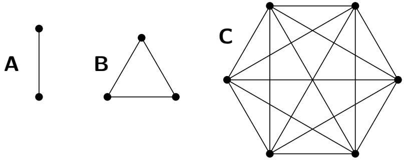

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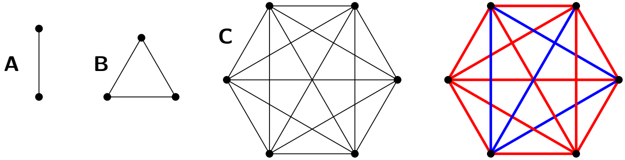
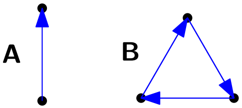


Order is necessary



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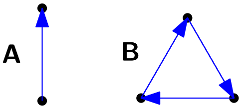
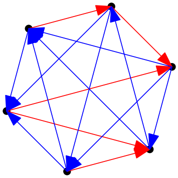


Vertices of **C** can be linearly ordered and edges coloured accordingly:

- If edge is goes forward in linear order it is red
- blue otherwise.



Ramsey classes

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Example (Partial orders — Nešetřil-Rödl, 84; Paoli-Trotter-Walker, 85)

The class of all finite partial orders with linear extension is Ramsey.

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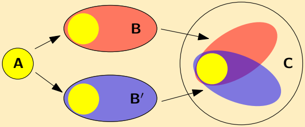
Example (Models — H.-Nešetřil, 2016)

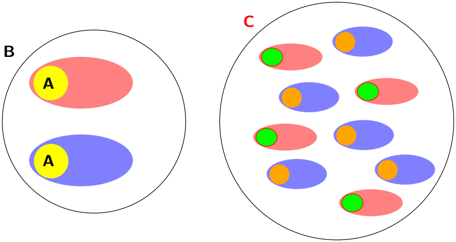
For every language L, $\overrightarrow{Mod}(L)$ is a Ramsey class.

Ramsey classes are amalgamation classes

Definition (Amalgamation)



Nešetřil, 80's: Under mild assumptions Ramsey classes have amalgamation property.



Nešetřil's Classification Programme, 2005

Classification Programme

Ramsey classes	$\Longrightarrow$	amalgamation classes
$\Uparrow$		$\Downarrow$
expansions of homogeneous	$\Longleftarrow$	homogeneous structures

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Let L' be language containing language L. A expansion (or lift) of L-structure $\mathbf{A}$ is L'-structure $\mathbf{A}'$ on the same vertex set such that all relations/functions in $L \cap L'$ are identical.

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Theorem (Nešetřil, 1989)

All homogeneous graphs have Ramsey expansion.

Nešetřil's Classification Programme, 2005

Classification Programme

amalgamation classes

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Nešetřil's Classification Programme, 2005

Classification Programme

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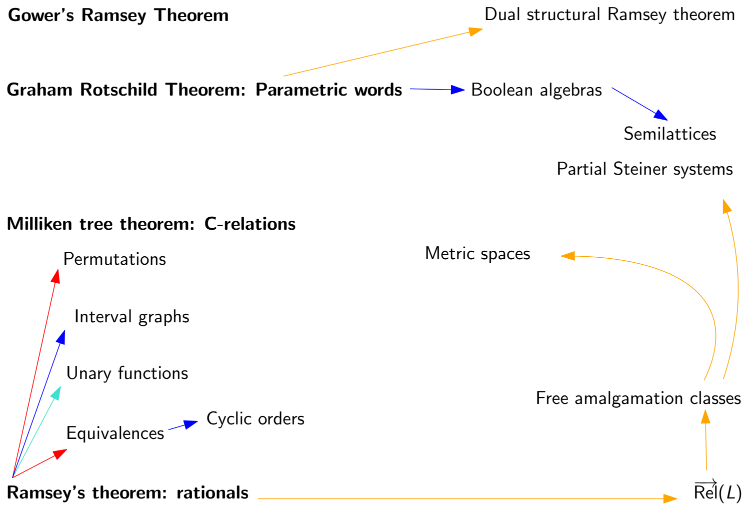
Free amalgamation classes

Ramsey's theorem: rationals → $\overrightarrow{\mathrm{Rel}}(L)$

Product arguments Interpretations Adding unary functions Partite construction



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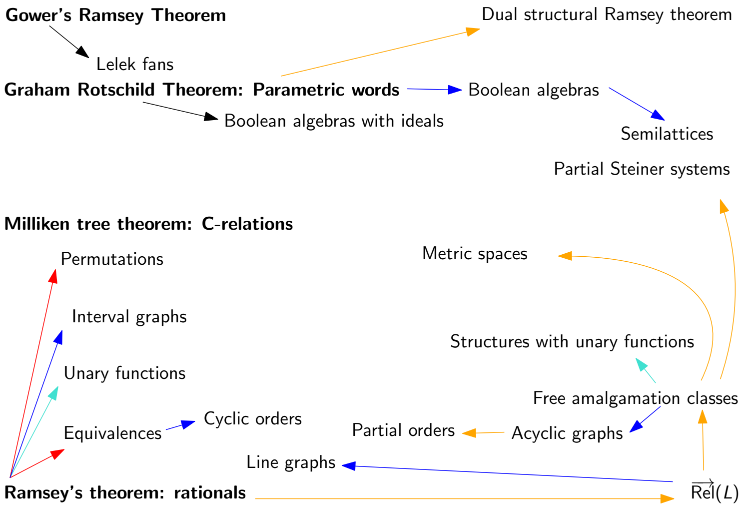

Product arguments   Interpretations   Adding unary functions   Partite construction

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Theorem (Kechris, Pestov, Todorčevič 2005, Nguyen van Thé 2012)

For every amalgamation class $\mathcal{K}$ there exists, up to bi-definability, at most one Ramsey class $\mathcal{K}'$ of precompact expansions of $\mathcal{K}$ with expansion property.

Existence of precompact expansions

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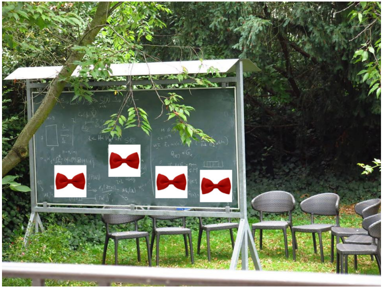

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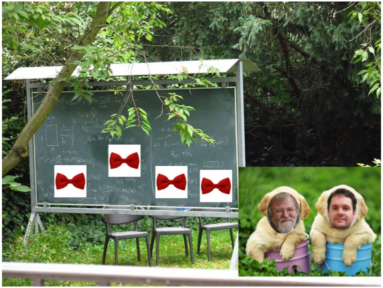
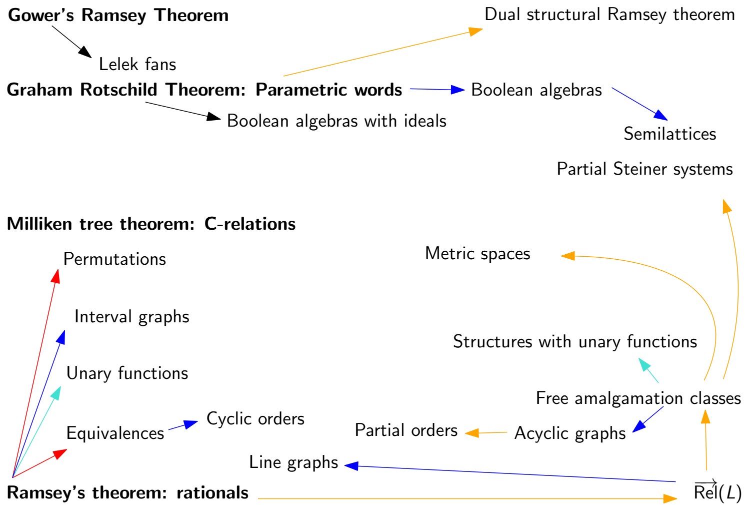
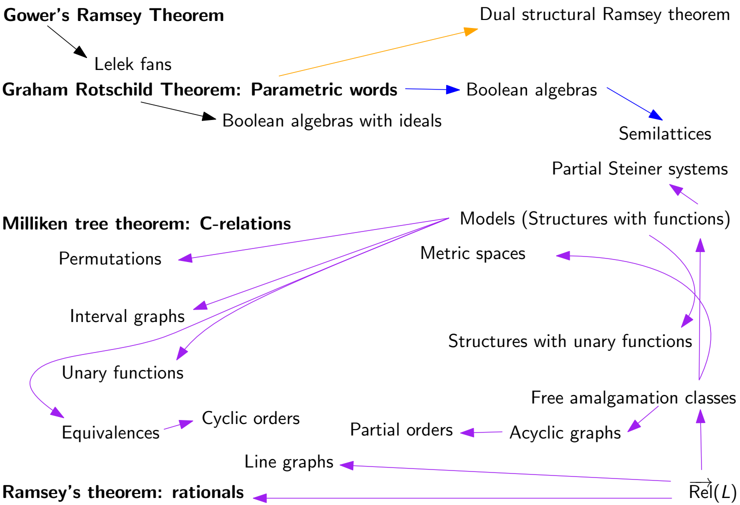




Product arguments | Interpretations | Adding unary functions | Partite construction



Locally finite subclass

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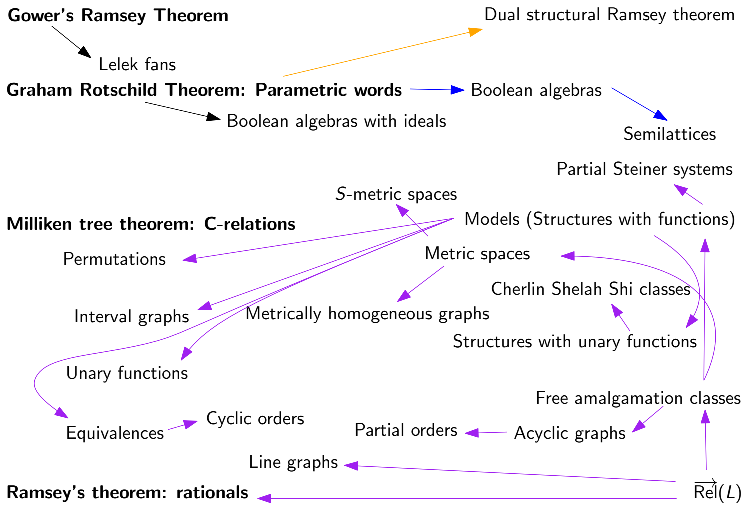

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Ramsey's theorem: rationals $\overrightarrow{\mathrm{Rel}}(L)$

Locally finite subclass

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Existence of precompact expansions

Theorem (Evans, 2015+)

There is a countable, ω-categorical structure $\mathbf{M}_F$ no precompact Ramsey expansion.

In this talk we explore properties of this example.

Three variants of David's example

- $\mathcal{C}_0$: The easy example
- $\mathcal{C}_1$: The kindergarten example
- $\mathcal{C}_F$: The actual counter-example

Hrushovski construction

- Predimension of a graph $\mathbf{G} = (V, E)$ is $\delta(G) = 2|V| - |E|$.

Example

$\delta(K_1) = 2$ $\quad\delta(K_2) = 4 - 1 = 3$ $\quad\delta(K_3) = 6 - 3 = 3$

$\delta(K_4) = 8 - 6 = 2$ $\quad\delta(K_5) = 10 - 10 = 0$ $\quad\delta(K_6) = 12 - 30 = -18.$

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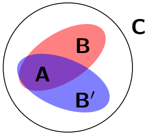
Lemma

$\mathcal{C}_0$ is closed for free amalgamation over self-sufficient substructures.

Proof.

$\delta(\mathbf{C}) = \delta(\mathbf{B}) + \delta(\mathbf{B}') - \delta(\mathbf{A})$. □



Hrushovski class $\mathcal{C}_0$ as a reduct

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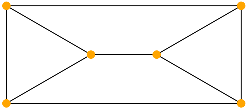

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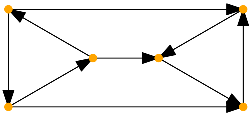

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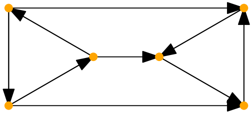

Corollary

$\mathcal{C}_0$ is a reduct of the class of all finite 2-orientations $\mathcal{D}_0$.

$\mathcal{D}_0$ is closed for free amalgamation over successor-closed substructures.

Ramsey expansions of $\mathcal{C}_0$ and orientations

Theorem (Kechris, Pestov, Todorčevič, 2005)

Let **F** *be a Fraïssé limit, then the following are equivalent.*

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Proof.

- Consider G acting on the space $X(M_0)$ of 2-orientations of $\mathbf{M}_0$ (a G-flow).
- As $\mathrm{Aut}(\mathbf{M}_0^+)$ is extremely amenable, there is some $\mathbf{S} \in X(\mathbf{M}_0)$ which is fixed by $\mathrm{Aut}(\mathbf{M}_0^+)$.
- $\mathrm{Aut}(\mathbf{M}_0^+)$ is a subgroup of $\mathrm{Aut}(\mathbf{S})$.

□

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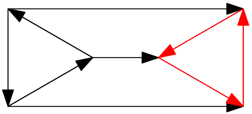

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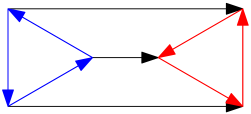

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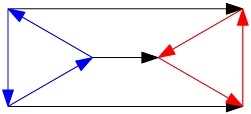

Proof.

- Every vertex $v \in \mathbf{M}_0^+$ has out-degree at most 2, but infinite in-degree.
- Oriented path $v_1 \to v_2 \to v_2 \ldots v_n$ always extendeds by a vertex v_0 to $v_0 \to v_1 \to v_2 \to v_2 \ldots v_n$.

□

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Denote by $\mathcal{D}_0^{\prec}$ the class of all finite ordered 2-orientations.

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Proof.

- Given $\mathbf{A}, \mathbf{B} \in \mathcal{D}_0^{\prec}$ put $N \longrightarrow (|B|)_2^{|A|}$.
- Extend language by unary predicates $R_1, R_2, \ldots R_N$.
- Given $|B|$ tuple $\vec{b} = (b_1, b_2, \ldots b_{|B|})$, denote by $\mathbf{B}_{\vec{b}}$ expansion of $\mathbf{B}$ where i-th vertex is in relation R_{b_i}.
- $\mathbf{P}_0$ is a disjoint union of $\mathbf{B}_{\vec{v}}$, $v \in \binom{n}{|B|}$.
- Put $u \sim v$ if successor-closure of u is isomorphic to v.
- $C = P_0/\sim$. $C \longrightarrow (B)_2^A$.

□

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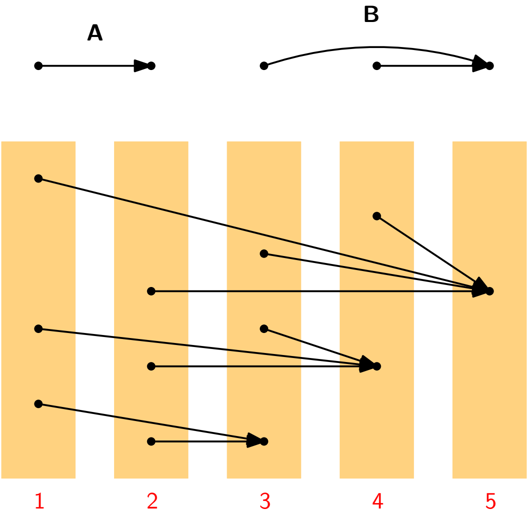

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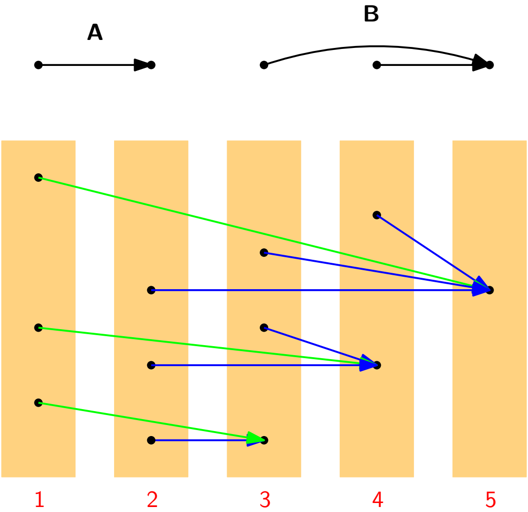

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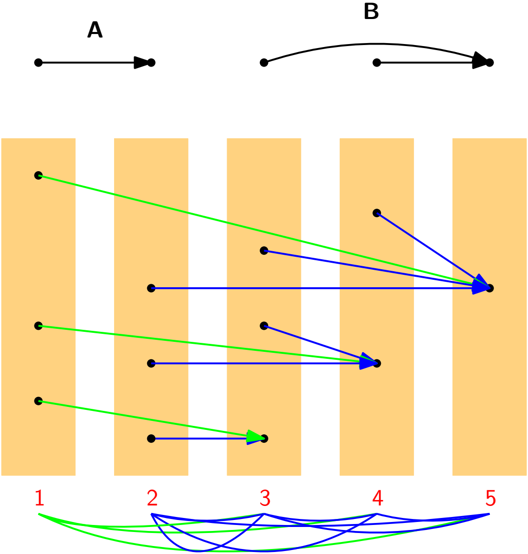

Optimality of Ramsey expansion

Question: (Tsankov)

Is $(\mathcal{D}_0^{\prec}; \sqsubseteq_s)$ any better than the trivial Ramsey expansion?

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Theorem (H., Evans, Nešetřil, 2016+)

There exists $\mathcal{G}_0 \subset \mathcal{D}_0^{\prec}$ such that

- $(\mathcal{G}_0; \sqsubseteq_s)$ *is strong expansion of* $(\mathcal{C}_0; \leq_s)$,
- $(\mathcal{G}_0; \sqsubseteq_s)$ *is Ramsey classes,*
- N_{G_0}, *the group of automorphisms of Fraïssé limit of* $(\mathcal{G}_0; \sqsubseteq_s)$ *is maximal amongst extremely amenable subgroups of* $\mathrm{Aut}(\mathbf{M}_0)$.
- *Class of all self-sufficient substructures of* $\mathcal{G}_0$ *has an Expansion Property with respect to* $\mathcal{C}_0$ *and thus give a minimal* $\mathrm{Aut}(\mathbf{M}_0)$ *flow.*

Expasion property of non-precompactness

Definition

$\mathcal{K}'$ has expansion property wrt $\mathcal{K}$ if for every $\mathbf{A} \in \mathcal{K}$ there exists $\mathbf{B} \in \mathcal{K}$ such that every expansion of $\mathbf{B}$ in $\mathcal{K}'$ contains every expansion of $\mathbf{A}$ in $\mathcal{K}'$.

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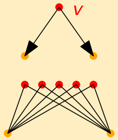

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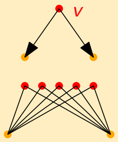

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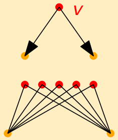

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- Construct $\mathbf{B}^0$ by induction hypothesis.
- Extend every copy of $\mathbf{A}^0$ in $\mathbf{B}^0$ to $\mathbf{A}$ by 5 copies of v.

□

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Suppose $\mathbf{A} \in \mathcal{D}_1$ we put $\mathbf{A} \in \mathcal{E}_1$ iff:

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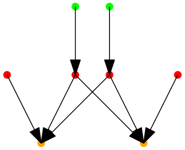

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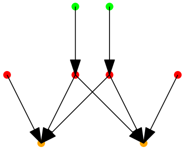

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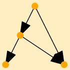

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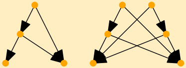

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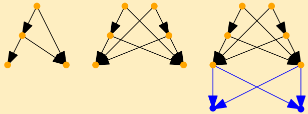

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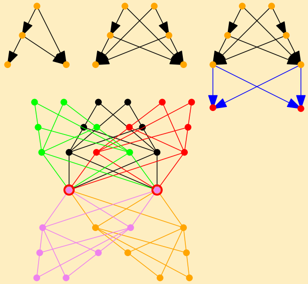

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Given strong class $(\mathcal{C}; \leq)$, a strong partial automorphism of $\mathbf{A} \in \mathcal{C}$ is an isomorphism $f : \mathbf{D} \to \mathbf{E}$ for some $\mathbf{D}, \mathbf{E} \leq \mathbf{A}$.

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- Put $(v, f) \sim (v, f')$ iff there is isomorphism of successor-closures α of (v, f) and (v, f') such that $\alpha(u, h) = (h, h')$.
- $\mathbf{B} = \mathbf{B}_0 / \sim$.

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Hrushovski construction has Hrushovski expansion

Theorem (H., Evans, Nešetřil, 2017+)

Let $\mathcal{D}$ be a class of finite 2-orientations closed for free amalgamation over successor-closed substructures. Then $\mathcal{D}$ has EPPA.

Proof.

- Given $\mathbf{A} \in \mathcal{D}$ construct $\mathbf{B}_0 \in \mathcal{D}$ as follows:
  1. Vertices of $\mathbf{B}_0$ are pairs (v, f) where $v \in \mathbf{A}$ and $f \in Sym(\mathbf{B})$.
  2. $(v, f) \to (v', f')$ iff $f = f'$ and $f(v) \to f(v')$ is edge of $\mathbf{A}$.
- Put $(v, f) \sim (v, f')$ iff there is isomorphism of successor-closures α of (v, f) and (v, f') such that $\alpha(u, h) = (h, h')$.
- $\mathbf{B} = \mathbf{B}_0 / \sim$.

□

Along with Herwig-Lascar theorem this also shows EPPA for unary Cherlin-Shelah-Shi classes and more.

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   ¬Ramsey, ¬EPPA, AP
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Put $F(x) = \ln(x)$. Then $(\mathcal{C}_F; \leq_d)$ is a free amalgamation class.

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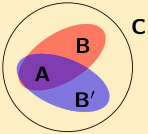

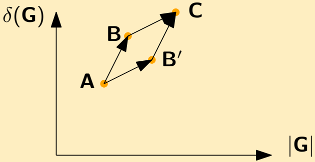

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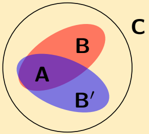

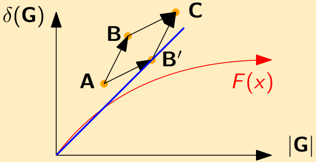

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$\text{roots}_{\mathbf{A}}(\mathbf{B})$ is set of all roots of **A** reachable from $\mathbf{B} \subseteq \mathbf{A}$

Lemma (H., Evans, Nešetřil, 2015+)

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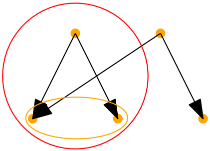

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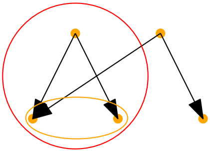
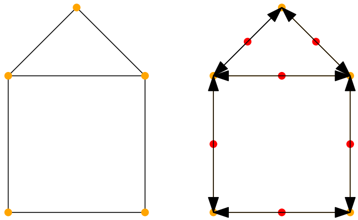


Proof.

- Given $\mathbf{B} \sqsubseteq_s \mathbf{A}$, $\delta(\mathbf{B})$ is the number of roots of out-degree 1 + twice number of roots of out-degree 0.
- Extending $\mathbf{B}$ by all vertices v such that $\mathrm{roots}_{\mathbf{A}}(v) \subseteq \mathrm{roots}_{\mathbf{A}}(\mathbf{B})$ keeps δ.
- Extending $\mathbf{B}$ by any other vertex increases δ.

□

$\mathcal{C}_F$ is harder



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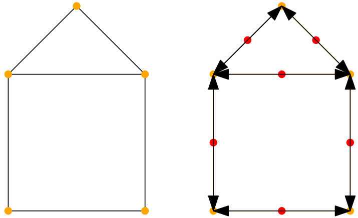

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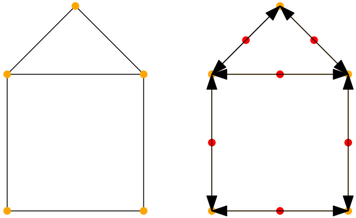

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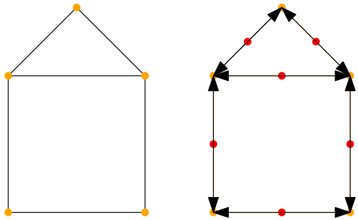

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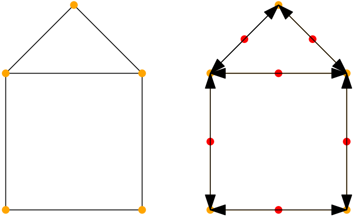

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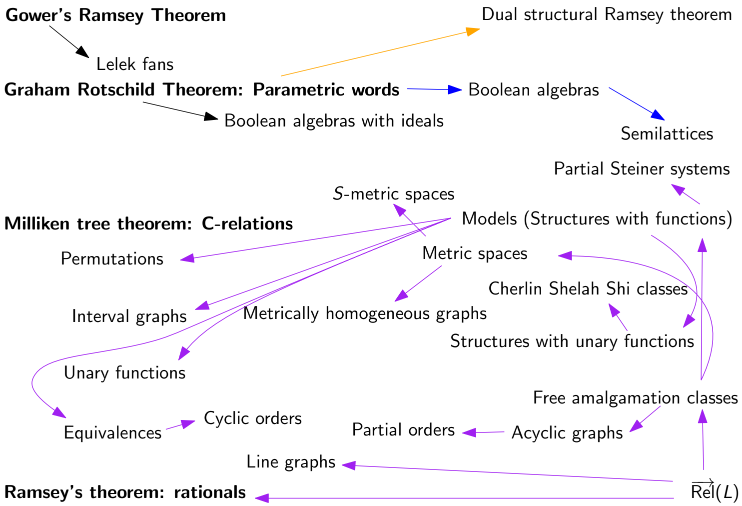
EPPA and big Ramsey degree currently open (WIP).



Locally finite subclass

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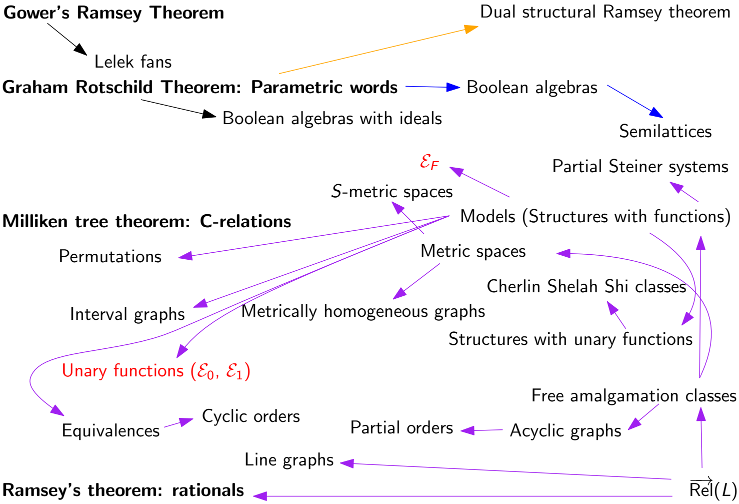

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Thank you for the attention

- J.H., J. Nešetřil: All those Ramsey classes (Ramsey classes with closures and forbidden homomorphisms). Submitted (arXiv:1606.07979), 2016, 60 pages.
- D. Evans, J.H., J. Nešetřil: Automorphism groups and Ramsey properties of sparse graphs. To appear soon, 53+ pages.