Structural Ramsey Theory

and

the Extension Property for Partial Automorphisms

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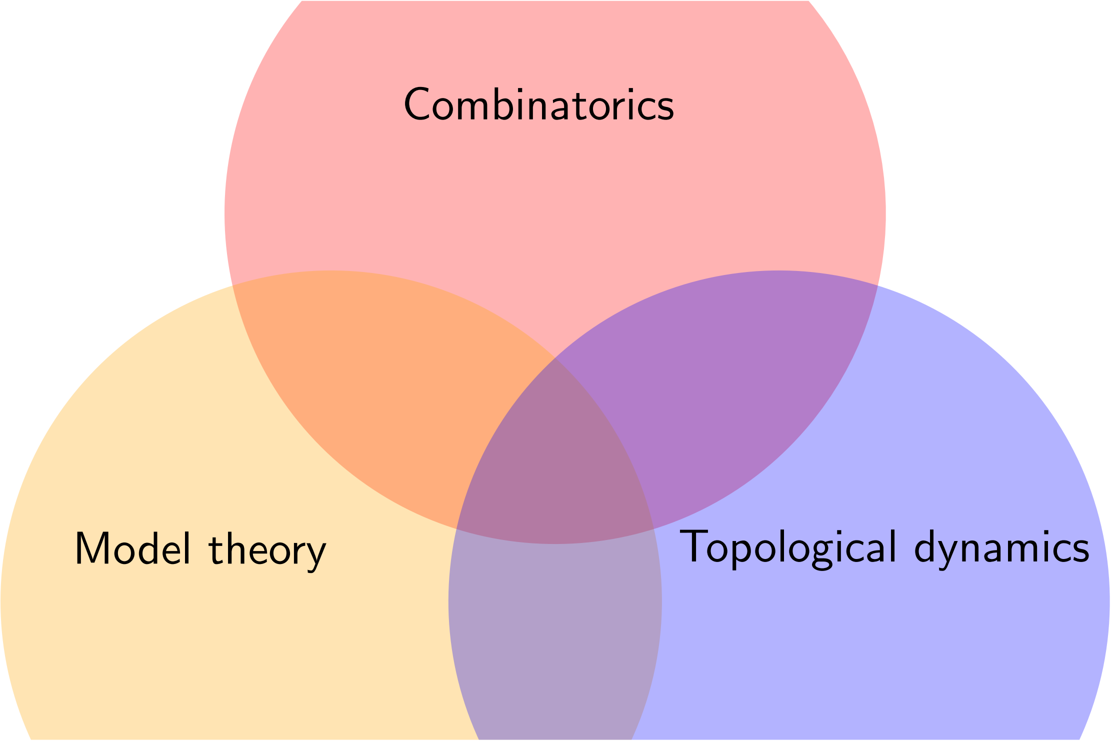

Combinatorics
Model theory
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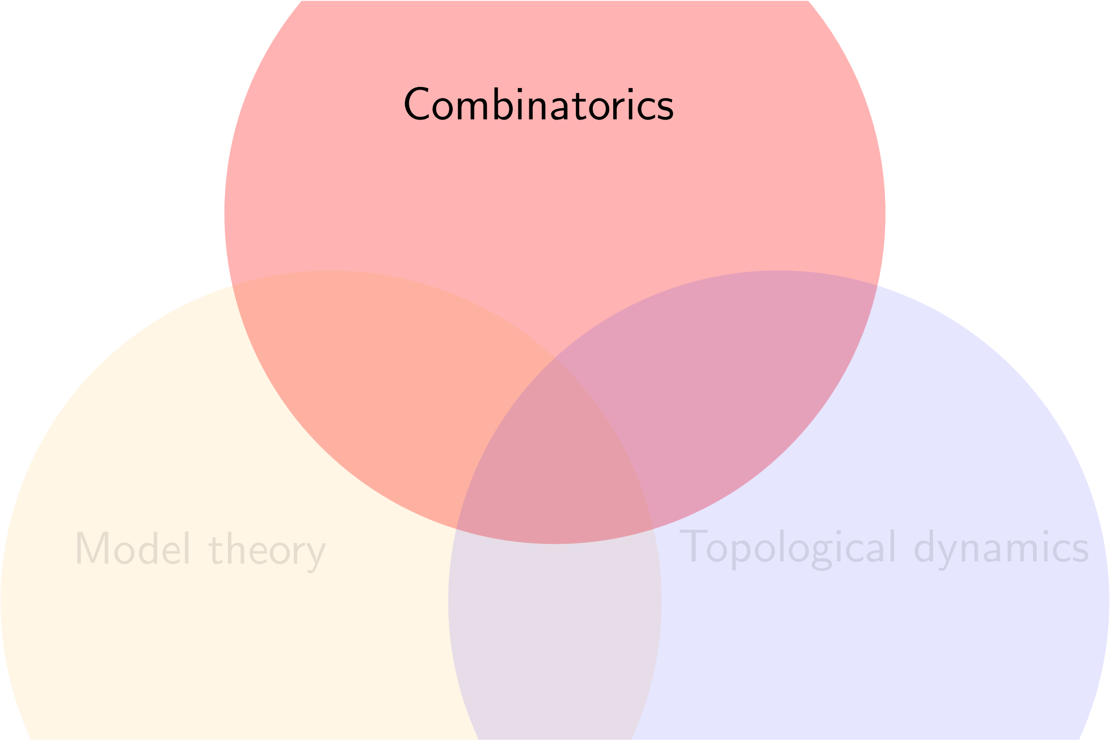
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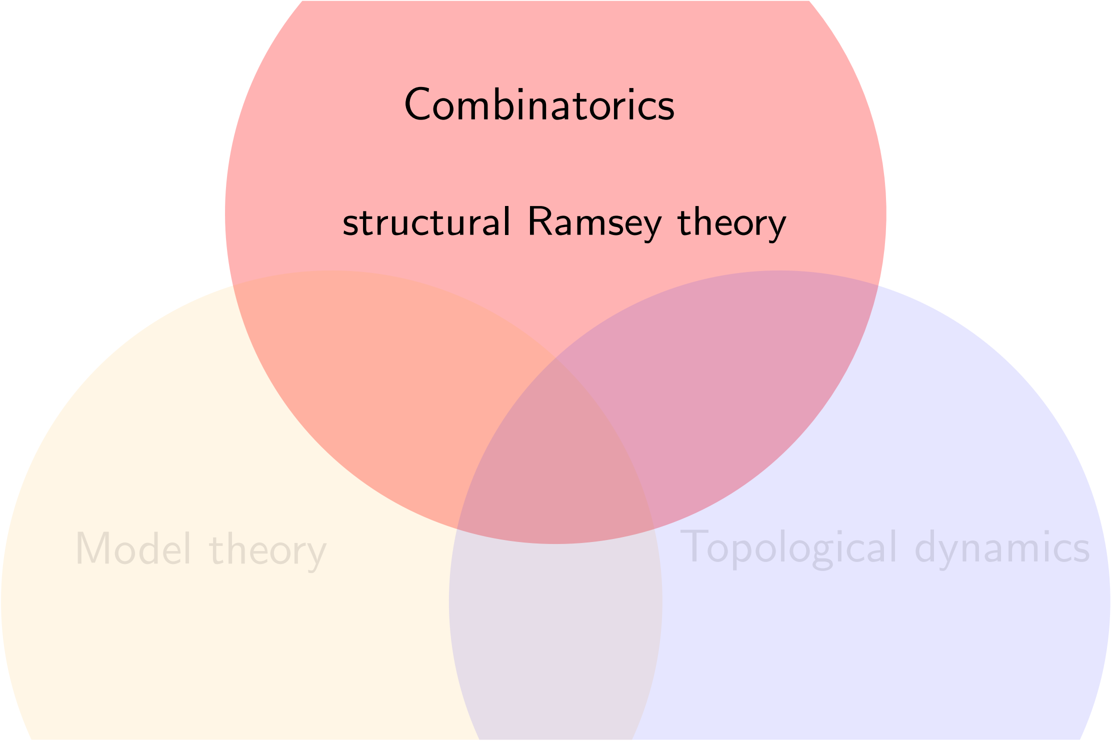
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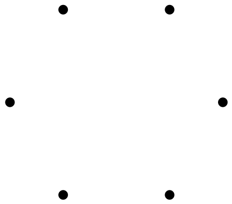

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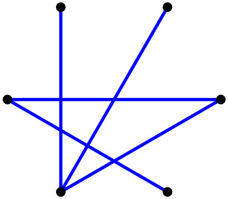

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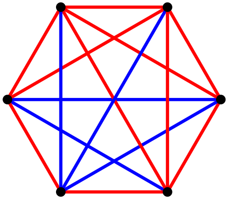

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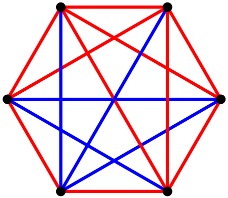

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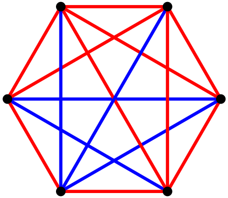

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Erdös–Rado partition arrow

$N \longrightarrow (n)^p_k$: For every partition of p-element subsets of X, $|X| \geq N$ into k classes (colours) there exists $Y \subseteq X$, $|Y| = n$ such that all p-element subsets of Y belongs to a single partition. (Y is monochromatic.)

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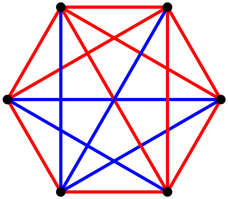
$p = 2, n = 3, k = 2, N = 6$



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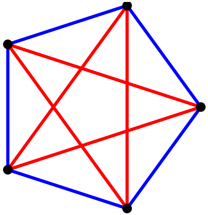

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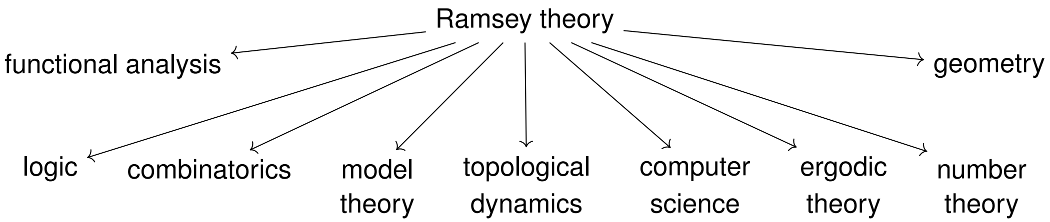
Many aspects of Ramsey theory



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For every pair of finite ordered graphs $\mathbf{A}$ *and* $\mathbf{B}$ *the exists a finite ordered graph* $\mathbf{C}$ *such that*

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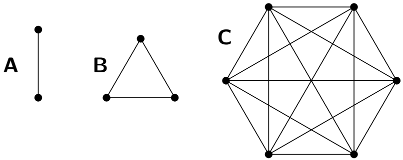

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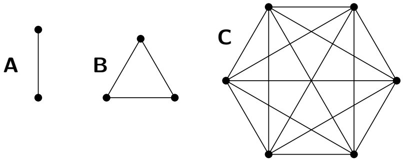

$\binom{\mathbf{B}}{\mathbf{A}}$ is the set of substructures of **B** isomorphic to **A**. (The set of all copies of **A** in **B**.)

$\mathbf{C} \longrightarrow (\mathbf{B})^{\mathbf{A}}_2$: For every 2-colouring of $\binom{\mathbf{C}}{\mathbf{A}}$ there exists $\widetilde{\mathbf{B}} \in \binom{\mathbf{C}}{\mathbf{B}}$ such that $\binom{\widetilde{\mathbf{B}}}{\mathbf{A}}$ is monochromatic.

Ramsey classes

Definition

A class $\mathcal{K}$ of finite structures is Ramsey iff for every $\mathbf{A}, \mathbf{B} \in \mathcal{K}$ there exists $\mathbf{C} \in \mathcal{K}$ such that

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Examples of Ramsey classes:

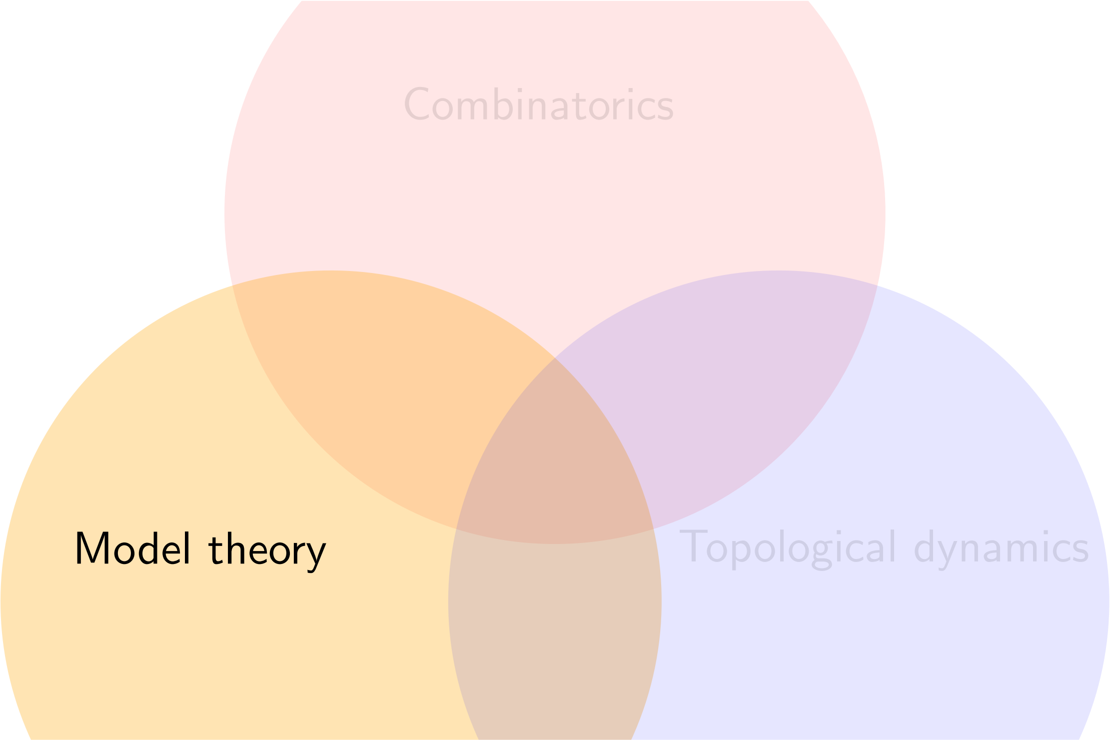
1. All finite linear orders
   (Ramsey theorem, 1930)
2. All finite ordered relational structures in a given language L
   (Nešetřil–Rödl, 1976; Abramson–Harrington, 1978)
3. Partial orders with linear extensions
   (Nešetřil–Rödl, 1984; Paoli–Trotter–Walker, 1985)
4. Ordered metric spaces
   (Nešetřil 2007)

New base structural Ramsey theorem

Theorem (H.–Nešetřil, 2019: Ramsey theorem for finite models)

For every language L, the class of all finite ordered structures in language L is Ramsey.

Language L can consist of relational symbols and function symbols.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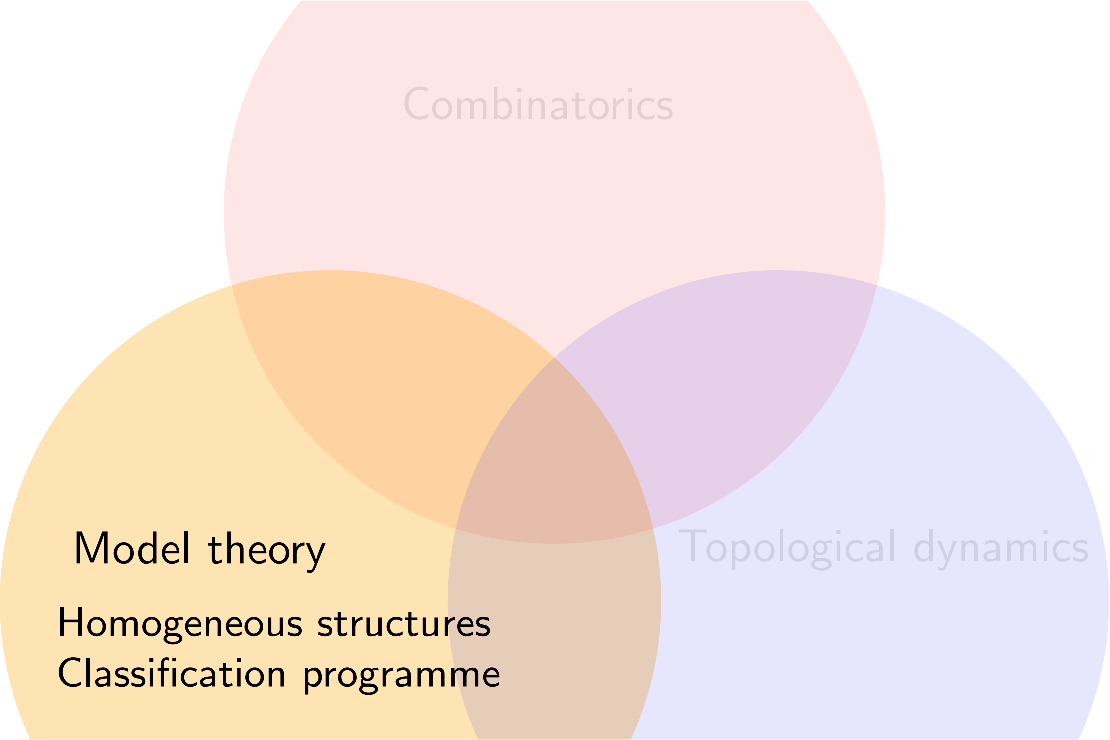
Combinatorics
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Homogeneous structures
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Homogeneous structures

Definition (Homogeneity)

Structure **H** is homogeneous if every isomorphism of its two finite (induced) substructures (a partial automorphism of **H**) extends to an automorphism of **H**.

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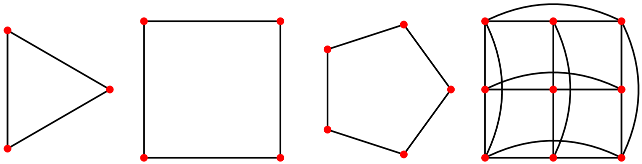

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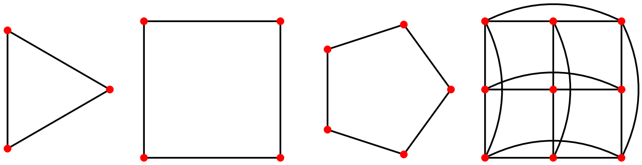
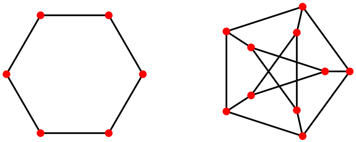


Non-examples:



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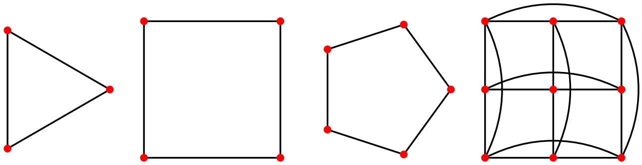

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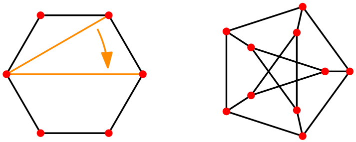

Amalgamation classes (Fraïssé theory)

Given structure **A** its age, Age(**A**), is the set of all finite structures with embedding to **A**.

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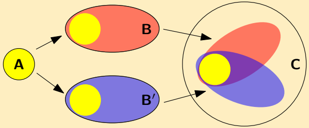
Class $\mathcal{K}$ of finite structures has the amalgamation property if for every $\mathbf{A}, \mathbf{B}, \mathbf{B}' \in \mathcal{K}$, and embeddings $\mathbf{A} \to \mathbf{B}$, $\mathbf{A} \to \mathbf{B}'$ there exists $\mathbf{C} \in \mathcal{K}$ satisfying:



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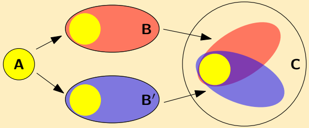

Theorem (Fraïssé, 1950s)

*A hereditary, isomorphism-closed class $\mathcal{K}$ with countably many mutually non-isomorphic structures is an age of a homogeneous structure **A** if and only if it has the amalgamation property.*

Nešetřil, 80's: Under mild assumptions Ramsey classes have amalgamation property.

Classification Programme

Cherlin–Lachlan's classification programme of homogeneous structures is a long running project providing full catalogues of homogeneous structures of a given type.
(Such as graphs, digraphs, . . .)

Homogeneous structure is Ramsey if its age is a Ramsey class.

Nešetřil's Classification Programme of Ramsey classes, 2005

$$\begin{array}{ccc} \text{Ramsey classes} & \Longrightarrow & \text{amalgamation classes} \\ \Uparrow & & \Downarrow \\ \text{Ramsey structures} & \Longleftarrow & \text{homogeneous structures} \end{array}$$

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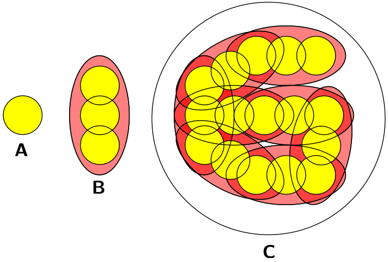
All homogeneous graphs and digraphs are known to have Ramsey expansions.

General structural conditions

Theorem (H.–Nešetřil 2019)

Let L be a language and $\mathcal{R}$ be a Ramsey class of L-structures. Then every *locally finite subclass* *of $\mathcal{R}$ with strong amalgamation property is Ramsey.*

Being a locally finite subclass is a new condition similar to the amalgamation property.



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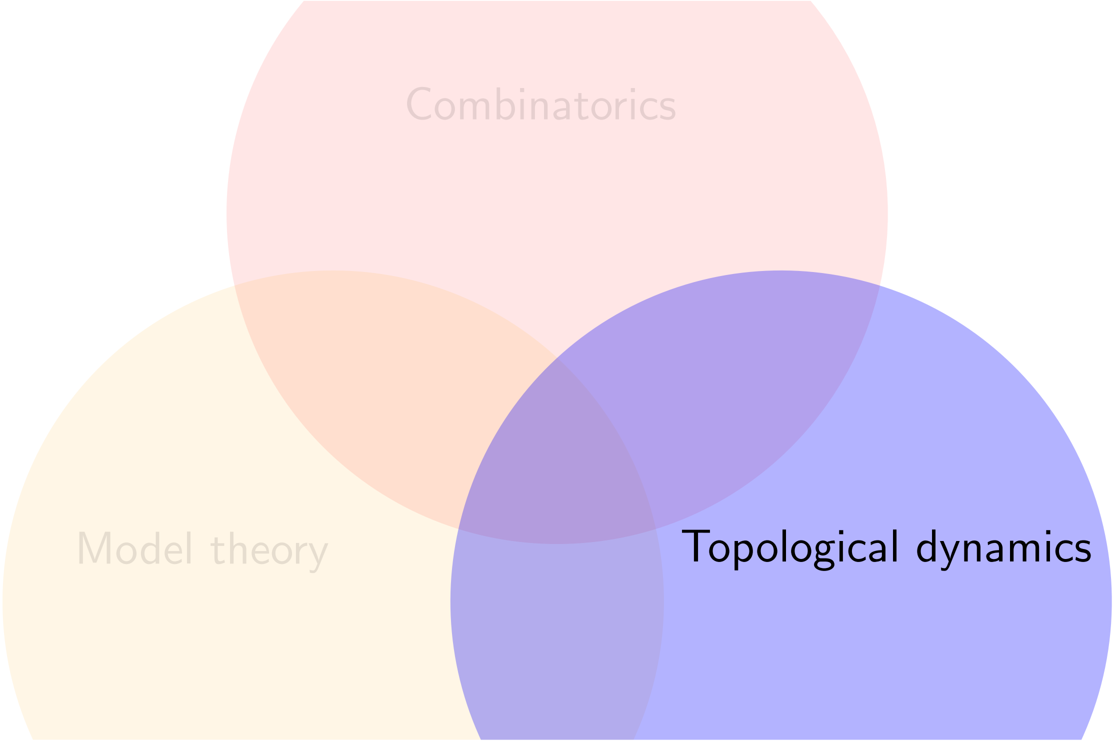
Then every locally finite subclass of $\mathcal{R}$ with strong amalgamation property is Ramsey.

This result implies the existence of a Ramsey expansion for all ages of homogeneous structures given by the classification programme.

Many additional application include all known metrically homogeneous graphs, structures defined by forbidden homomorphisms, . . .

H., J. Nešetřil. All those Ramsey classes (Ramsey classes with closures and forbidden homomorphisms). Advances in Mathematics (2019). 89p. (46 citations by Google scholar)

Aranda, Bodirsky, Bradley–Williams, Brady, Braunfeld, Coulson, Evans, Karamanlis, Kompatcher, Konečný, Madelaine, Motett, Pawliuk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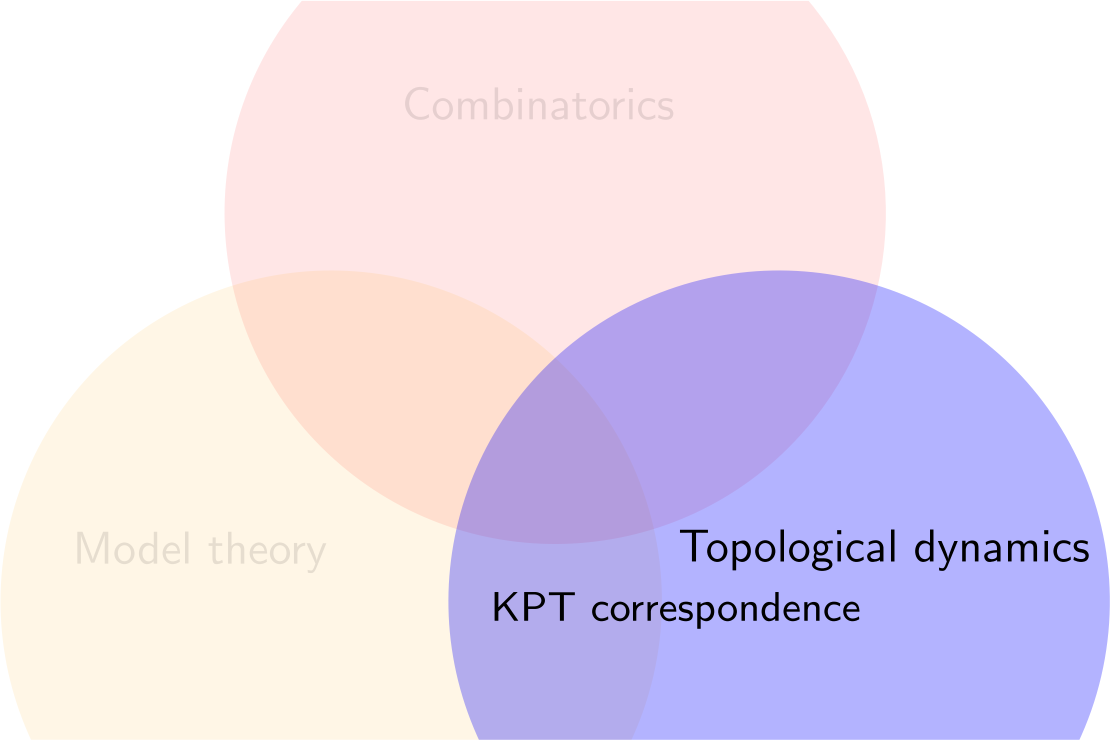
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KPT correspondence

Kechris, Pestov, Todorčevič correspondence

Nešetřil's Classification Programme of Ramsey classes

Ramsey classes	$\Longrightarrow$	amalgamation classes
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$\Downarrow \Uparrow$		$\Downarrow$
extremely amenable groups	$\Longrightarrow$	universal minimal flows

Theorem (Kechris, Pestov, Todorčevič 2005: KPT-correspondence)

The group of automorphisms of the Fraïssé limit of a amalgamation class $\mathcal{K}$ is extremely amenable if and only if $\mathcal{K}$ is a Ramsey class.

Conjecture (Nguyen Van Thé, 2013)

Every ω-categorical structure has a precompact Ramsey expansion with the expansion property.

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Theorem (Evans–H.–Nešetřil, 2019)

There is a countable, ω-categorical structure $\mathbf{M}$ *with no precompact Ramsey expansion.*

Proved using the KPT-correspondence and Hrushovski predimension construction (an advanced tool of model theory).

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Question (Bodirsky, Pinsker, Tsankov, 2011)

Does every homogeneous structure in a finite relational language have a precompact Ramsey expansion?

The question is motivated by applications in the area of infinite-domain constraint satisfaction problems (CSP)

Applications — broader perspectives

1. Group theory
   (extension property for partial automorphisms — EPPA)
2. Infinitary Ramsey theory
   (big Ramsey degrees)

Extension property for partial automorphisms (EPPA)

Definition (Extension property for partial automorphisms)

A class $\mathcal{K}$ of finite structures has extension property for partial automorphisms (EPPA) iff for every $\mathbf{A} \in \mathcal{K}$ there exists $\mathbf{B} \in \mathcal{K}$ containing $\mathbf{A}$ such that every partial automorphism of $\mathbf{A}$ extends to an automorphism of $\mathbf{B}$.

Main contributions:

1. EPPA theorem for finite models: Γ_L-structures with relations and unary functions (H., Konečný, Nešetřil, 2020+, motivated by a construction by Hodkinson and Otto)
2. Satisfactory structural condition for a class to have EPPA (H., Konečný, Nešetřil, 2020+)
3. EPPA for two-graphs and antipodal metric spaces and for semigeneric tournaments (Evans, H., Konečný, Nešetřil 2020; Jahel, H., Konečný, Sabok, 2020+)
4. A non-trivial example of homogeneous structure with no precompact EPPA expansion (Evans, H., Nešetřil, 2019)

Infinitary Structural Ramsey theory

Theorem (Laver 1969, published 1984)

The order of rationals has finite big Ramsey degrees.

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Additional known structures with finite big Ramsey degrees:

1. Rado graph (Sauer, 2005)
2. Universal homogeneous triangle-free graph (Dobrinen, 2019)
3. Henson graphs (Dobrinen, 2020+)
4. Fraïssé limits of free amalgamation classes in binary language (Zucker 2020+)

Infinitary Structural Ramsey theory

Theorem (H. 2020+)

The countable universal homogeneous partial order has finite big Ramsey degrees.

Proved by techniques inspired by the EPPA construction applying Carlson–Simpson theorem.

Theorem (Balko, Chodounský, H., Konečný, Vena, 2019+)

For every finite relational language L the universal homogeneous structure in language L has finite big Ramsey degrees.

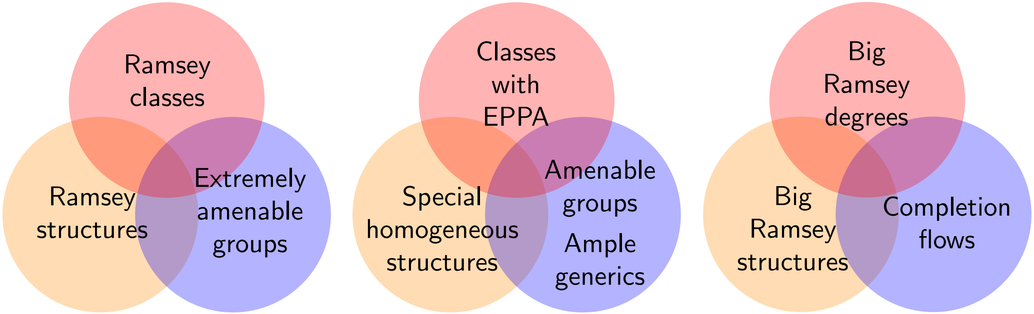
Open problems

Open problems:

1. Ramsey expansions of the class of all finite graphs of girth $g \geq 4$.
2. Ramsey expansions of the class of all partial Steiner systems omitting odd cycles of length at most ℓ.
3. Does the class of all finite partial Steiner systems have EPPA?
4. EPPA for classes with bounded of equivalence relations on pairs of vertices.
5. Are big Ramsey degrees of the universal homogeneous 3-uniform hypergraph omitting complete subhypergraph on 6 vertices finite?
6. Characterise big Ramsey degrees of partial orders, metric spaces, ...

General directions:

1. Bring understanding of all three areas to the same level.
2. Develop necessary and sufficient structural conditions for a given class to be Ramsey, to have EPPA or its Fraïssé limit to have finite big Ramsey degrees
3. Develop a general theory covering all three phenomena



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Thank you

Papers included in thesis

1. D. M. Evans, H., J. Nešetřil. Ramsey properties and extending partial automorphisms for classes of finite structures. To appear in Fund. Math. (2020). 41p.
2. D. M. Evans, H., M. Konečný, J. Nešetřil. EPPA for two-graphs and antipodal metric spaces. Proc. Amer. Math. Soc. 148.5 (2020): 1901–1915.
3. H., M. Konečný, J. Nešetřil. All those EPPA classes (strengthenings of the Herwig–Lascar theorem). arXiv:1902.03855, submitted (2020). 44p
4. A. Aranda, D. Bradley–Williams, J. H., M. Karamanlis, M. Kompatcher, M. Konečný, M. Pawliuk. Ramsey expansions of metrically homogeneous graphs. European J. Combin, accepted. 57p.
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