Explicit construction of universal structures

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Joint work with Jarik Nešetřil

Workshop on Homogeneous Structures 2011
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Let $C$ be class of relational structures.

**Definition**

Relational structure $U$ is *(embedding-)universal* for class $C$ iff $U \in C$ and every structure $A \in C$ is induced substructure of $U$. 
Class: graphs
Example

- **Class:** graphs
- **Universal graph:**
  - **Fraïssé:** homogeneous universal graph constructed by Fraïssé limit.
  - **Erdős and Rényi, 1963:** The countable random graph.
  - **Rado, 1965:** Explicit description:
    - **Vertices:** all finite 0–1 sequences \((a_1, a_2, \ldots, a_t), t \in \mathbb{N}\)
    - **Edges:** \(\{(a_1, a_2, \ldots, a_t), (b_1, b_2, \ldots, b_s)\}\) form edge if
      \[b_a = 1 \text{ where } a = \sum_{i=1}^{t} a_i 2^i.\]
Example

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      \(\iff b_a = 1\) where \(a = \sum_{i=1}^{t} a_i2^i\).
    - Many variants of Rado’s description are known.
    - All the description give up to isomorphism unique graph, as can be shown using the extension property.
Class: linear orders

Universal structure: $\mathbb{Q}$. 
Universal partial order

- **Class**: partial orders
- Homogeneous universal partial order exists by Fraïssé.

**Sketch of explicit description** (H., Nešetřil, 2003):

**Notation:**

Pairs $M = (M_L, M_R)$.

- $M_L$, $M_R$ are sets.

**Vertices:**

Pair $M$ is a vertex iff:

1. (left completeness) $A_L \subseteq M_L$ for each $A \in M_L$,
2. (right completeness) $B_R \subseteq M_R$ for each $B \in M_R$,
3. (correctness) 1. Elements $M_L$ and $M_R$ are vertices, 2. $M_L \cap M_R = \emptyset$,
4. (ordering property) $(\{A\} \cup A_R) \cap (\{B\} \cup B_L) \neq \emptyset$ for each $A \in M_L$, $B \in M_R$.

**Relation:**

We put $M < N$ if $(\{M\} \cup M_R) \cap (\{N\} \cup N_L) \neq \emptyset$.

Correspondence to Conway's surreal numbers. Later generalized to rational metric space (in H., Nešetřil, 2008; in constructive setting Lešnik, 2008).
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  - Correspondence to Conway’s surreal numbers.
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Cameron’s question

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Answer: I don’t know of any.
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However there are positive examples of universal partial order.
Definition

\(\{0, 1\}^*\) denote all words over alphabet \(\{0, 1\}\).

\(W \leq_w W'\) iff \(W'\) is an initial segment (left factor) of \(W\).
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\( W \leq_w W' \) iff \( W' \) is an initial segment (left factor) of \( W \).

Partial order \((\mathcal{W}, \leq_w)\):

**Vertices:** finite subsets \( A \) of \( \{0, 1\}^* \) such that no distinct words \( W, W' \) in \( A \) satisfy \( W \leq_w W' \).

**Relation:** \( A, B \in \mathcal{W} \) we put \( A \leq \mathcal{W} B \) when for each \( W \in A \) there exists \( W' \in B \) such that \( W \leq_w W' \).
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Is it homogeneous?
## Word order

### Definition

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- **Is it homogeneous?**
  - no: \(A = \{0\}, B = \{00, 01\}\) form a gap.
Lemma (H., J. Nešetřil, 2011)

$(\mathcal{W}, \leq_{\mathcal{W}})$ is an universal partial order

- We give an algorithm for on-line embedding of any partial order into $(\mathcal{W}, \leq_{\mathcal{W}})$.
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- Alice-Bob game:
  - Bob choose arbitrary partial order on vertices \(\{1, 2, \ldots N\}\).
  - At turn \(n\) Bob reveals the relations of vertex \(n\) to vertices \(1, 2, \ldots n - 1\).
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- **Sample game:**
  - **Alice:** Representation of 1 is $\{0\}$.
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- Sample game:
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  - **Alice:** Representation of 3 is \(\{000, 100\}\).
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  - **Alice**: Representation of 4 is \(\{0000\}\).
- We prove by induction that there is winning strategy for Alice. Basic idea is “Venn diagram” property.
Theorem (H., Nešetřil, 2004)

*The quasi order formed by finite oriented paths ordered by homomorphisms contains universal partial order.*

**Proof (sketch)**

Embed $(W, \leq W)$ into homomorphism order. Assign every word $W$ a path $P(W)$ such that $W \leq W'$ iff $P(W) \rightarrow P(W')$. Paths consist of head $H$, bodies $B_0, B_1$, and the tail $T$. For every set of words $A \in W$, $P'(A)$ is the disjoint union of paths $P(W), W \in A$.

Observation: $P'(A) \leq P'(B)$ iff $A \leq W B$.

*Little problem*: How to glue disjoint paths into single path?

New proof (H., Nešetřil, 2011) by embedding periodic sets of natural numbers.
Universality by embedding

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- For every set of words $A \in \mathcal{W}$, $P'(A)$ is disjoint union of paths $P(W), W \in A$. 
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Explicit construction of universal structures
Theorem (H., Nešetřil, 2004)

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Proof (sketch)

- Embed $\langle \mathcal{W}, \leq_\mathcal{W} \rangle$ into homomorphism order.
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- Path consist of head $H$, bodies $B_0, B_1$ and the tail $T$.
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Catalogue of universal partial orders

Order implied by clones on boolean functions, 7.3

Homomorphism orders of special classes of structures, 7.2

Convex sets \((C, \leq_C), 6.4\)

Homomorphism order of oriented paths \((P, \leq_P), 7.1\)

Piecewise linear functions \((F, \leq_F), 6.4\)

Truncated vectors \((TV, \leq_{TV}), 6.6\)

Order implied by grammars \((G, \leq_G), 6.5\).

Sets of intervals \((I, \leq_I), 6.3\)

Periodic sets \((S, \subseteq), 6.7\)

Binary tree dominance \((B, \leq_B), 6.2\)

Words \((W, \leq_W), 6.1\)
Universal but not homogeneous

- **Hajnal, Pach, 1981:**
  - Nonexistence of universal 4-cycle-free graph.

- **Komjáth, Mekler, Pach, 1988:**
  - Existence of universal graph $\mathcal{P}_l$-free graph ($\mathcal{P}_l$ is graph of length $l$).
  - Existence of universal graph for classes without short odd cycles (fixed proof appears in 1999).

- **Covington, 1989:**
  - Existence of universal graph for class of graphs without induced path on 4 vertices.
  - Notion of amalgamation failure.

- **Komjáth, 1999:**
  - Existence of universal bowtie-free graph.

- **Cherlin, Shelah, Shi, 1999:**
  - Characterization of universal $\omega$-categorical $\mathcal{F}$-free graphs via algebraic closure.
  - Existence of universal graph for classes defined by forbidden homomorphisms.
  - New examples

Explicit construction of universal structures
Universal graph without odd cycles of length at most 2$l$ + 1

*M-Structure* is structure $\mathbf{M} = (V, G, F_1, \ldots, F_{2l+1})$ such that:

1. $G$ is graph on $V$ without loops;
2. $F_1, \ldots F_{2l+1}$ graph on $V$ with loops;
3. $F_1 = G$;
4. $xy \in F_a, yz \in F_b, a + b \leq 2s + 1$, then $xz \in F_{a+b}$;
5. if $a + b \leq 2s + 1$ odd, then $F_a \cup G_b = \emptyset$.

**Universal graph:**

- Retract of Fraïssé limit of $M$ structures.
Universal graph without odd cycles of length at most $2l + 1$

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Universal graph:

- Retract of Fraïssé limit of $M$ structures.
- Retract of generic even-odd metric space with forbidden loop of length $\leq 2l + 1$.
- Metric graph
Covington’s construction of a universal structure for class $C$:

1. Identification of finite set of amalgamation failures
2. Extending language by new relations (homogenization), class $C'$
3. Universal structure is then reduct of the Fraïssé limit of $C'$
Universal graph with forbidden induced path on 4 vertices

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Amalgamation failures:
Universal graph with forbidden induced path on 4 vertices

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Amalgamation failures:

Language of graphs needs to be extended by single ternary relation.
A family of connected finite relational structures. Class $\text{Forb}_h(\mathcal{F})$ consists of all relational structures $A$ such that there is no homomorphism $F \rightarrow A$, $F \in \mathcal{F}$.

**Corollary (Cherlin, Shelah, Shi 1999)**

*There is universal graph for class $\text{Forb}_h(\mathcal{F})$.*

Proof by finiteness of the algebraic closure.
A family of connected finite relational structures. Class $\text{Forb}_h(\mathcal{F})$ consists of all relational structures $A$ such that there is no homomorphism $F \to A, F \in \mathcal{F}$.

**Corollary (Cherlin, Shelah, Shi 1999)**

*There is universal graph for class $\text{Forb}_h(\mathcal{F})$.*

Proof by finiteness of the algebraic closure.

Cherlin, Shelah, Shi give an condition on existence of universal $\omega$ categorical structure for $\mathcal{F}$-free graphs.
Explicit amalgamation argument for existence of universal graph for $Forb_h(F)$

**Definition**

For relational structure $A$ and inclusion minimal vertex cut $C$ in its Gaifman graph of $A$, a *piece of relational structure* $A$ is pair $P = (P, C)$. Here $P$ is structure induced on $A$ by union of $C$ and vertices of some connected component of $A \setminus C$.

Tuple $C$ consist of the vertices of cut $C$ in (arbitrary) linear order.

Vertices $C$ are *roots of piece* $P$. 
Examples

- The pieces of Petersen graph
The pieces of Petersen graph

Pieces of cycles of length $n = \text{paths of length } 2, \ldots, n - 2$ rooted at both ends.
Examples

- Pieces of a relational tree $T = \text{branches of } T$. 

\begin{center}
\begin{tikzpicture}
  \node at (0,0) (A) {A};
  \node at (-1.732,1) (B) {B};
  \node at (1.732,1) (C) {C};
  \draw (A) -- (B);
  \draw (A) -- (C);
  \draw (B) -- (C);
  \node at (3,0) (A1) {A};
  \node at (4.732,1) (B1) {B};
  \node at (2.268,1) (C1) {C};
  \draw (A1) -- (B1);
  \draw (A1) -- (C1);
  \draw (B1) -- (C1);
\end{tikzpicture}
\end{center}
Proof (sketch)

- Enumerate all pieces of all forbidden structures \( F \in \mathcal{F} \) as \( P_1 = (P_1, \vec{C}_1), \ldots, P_N = (P_N, \vec{C}_N) \).
Proof (sketch)

- Enumerate all pieces of all forbidden structures $F \in \mathcal{F}$ as $P_1 = (P_1, \vec{C}_1), \ldots, P_N = (P_N, \vec{C}_N)$.

- **Expansion $R'$ of structure $R \in Forb_h(\mathcal{F})$:**
  For every piece $P_i, i = 1, 2, \ldots, N$ add new relation $X_i$ of arity $\vec{C}_i$. 

Existence of homomorphism $f$: $P_i \rightarrow R$ imply $f(\vec{C}_i) \in X_i$.

Let $P_{i1}, \ldots, P_{in}$ be all pieces generated by cut $\vec{C}' \in X_{i1}, \ldots, C' \in X_{in}$.

There is no tuple $\vec{T}$ of vertices of $R$ such that $\vec{T} \in X_{i1}, \ldots, C' \in X_{in}$.

Substructures of expansions of all $R \in Forb_h(\mathcal{F})$ form an amalgamation class.

Reduct of the Fraïssé limit of this class is an universal graph for $Forb_h(\mathcal{F})$. 

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  For every piece $\mathcal{P}_i, i = 1, 2, \ldots, N$ add new relation $X_i$ of arity $\vec{C}_i$.
  
  - Existence of homomorphism $f : P_i \rightarrow R$ imply $f(\vec{C}_i) \in X_i$.
  - Let $\mathcal{P}_{i_1}, \ldots \mathcal{P}_{i_n}$ be all pieces generated by cut $\vec{C}$.
    There is no tuple $\vec{T}$ of vertices of $R$ such that $\vec{T} \in X_{i_1}, \ldots C' \in X_{i_n}$.
Enumerate all pieces of all forbidden structures $F \in \mathcal{F}$ as $\mathcal{P}_1 = (P_1, \vec{C}_1), \ldots, \mathcal{P}_N = (P_N, \vec{C}_N)$.

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- Existence of homomorphism $f : P_i \to R$ imply $f(\vec{C}_i) \in X_i$.
- Let $\mathcal{P}_{i_1}, \ldots \mathcal{P}_{i_n}$ be all pieces generated by cut $\vec{C}$.
  There is no tuple $\vec{T}$ of vertices of $R$ such that $\vec{T} \in X_{i_1}, \ldots C' \in X_{i_n}$.

Substructures of expansions of all $R \in \text{Forb}_h(\mathcal{F})$ form an amalgamation class. 
Reduct of the Fraïssé limit of this class is an universal graph for $\text{Forb}_h(\mathcal{F})$. 

Jan Hubička
Explicit construction of universal structures
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Study of specific examples

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- **Relations $F$** such that their Gaifman graph is simple:
  - Forbidden irreducible structures.
  - No finite homomorphism universal object
  - Blown up finite graph, with forbidden cliques.
Study of specific examples

- Arity 2: forbidden cycles, etc.
  - Representation translate to a metric space
  - Explicit construction of metric space can be directly used to represent these.

- Beyond arity 2 explicit representation still possible, but impractical to describe in full generality.
Thank you... 

...Questions?