Topics in Advanced Combinatorics
Lecture 1 (summary)

A k-coloring of a graph G is a mapping $c : V(G) \to \{1, \ldots, k\}$ such that $c(u) \neq c(v)$ for every $uv \in E(G)$. The smallest k such that a graph G is k-colorable, i.e., G has a k-coloring, is the chromatic number of G and is denoted by $\chi(G)$. A graph G is bipartite if it is 2-colorable. The chromatic number of every graph is at least its clique number, i.e., the number of vertices of the largest complete subgraph, and at most the maximum degree of G plus one. One of the most famous results in graph theory is the Four Color Theorem, which asserts that the chromatic number of every planar graph is at most four. Recall that a graph G is planar if G can be drawn in the plane without edges crossing.

We will be interested in a generalization of the concept introduced by Erdős, Rubin and Taylor. Consider a graph G. A k-list-assignment is a function $L : V(G) \to 2^{\mathbb{N}}$ such that $|L(v)| = k$ for every $v \in V(G)$. A graph is said to be k-list-colorable if for every k-list-assignment L, there exists $c : V(G) \to \mathbb{N}$ such that $c(v) \in L(v)$ for every $v \in V(G)$ and $c(u) \neq c(v)$ for every $uv \in E(G)$. The smallest k such that a graph G is k-list-colorable is the list chromatic number of G and is denoted by $\chi_\ell(G)$. Observe that $\chi(G) \leq \chi_\ell(G)$ for every graph G and $\chi_\ell(K_{3,3}) > 2$, i.e., the chromatic number and the list chromatic may differ.

The following were two main results presented during the lecture.

Theorem. *There exists a planar graph with list chromatic number at least five.*

Theorem. *Every planar graph has list chromatic number at most five.*

Exercises

1. Show that the list chromatic number of $K_{3,3}$ is equal to 3.

2. For every k, find a bipartite graph with list chromatic number at least k.

3. Find a bipartite planar graph with list chromatic number at least 3.

4. Find a triangle-free planar graph with list chromatic number at least 4.