Topics in Advanced Combinatorics
Lecture 2 (summary)

A graph is planar if it can be embedded in the plane. A plane graph is a graph with a fixed embedding in the plane. Edge-maximal plane graphs are triangulations as asserted by the next lemma.

Lemma. *Every plane graph with at least three vertices is a spanning subgraph of a plane triangulation.*

Euler's Formula asserts that every connected plane graph G satisfies that $n+f=m+2$ where n is the number of vertices of G, f is the number of faces and m is the number of edges. Euler's Formula yields the following upper bound on the number of edges of a plane graph.

Lemma. *Every n-vertex planar graph, $n \geq 3$, has at most $3n-6$ edges.*

We now consider embedding of graphs to surfaces of higher genera, which are obtained from the plane by adding g handles; in this way, we obtain a surface of Euler genus g. Euler's Formula can be extended to surfaces of higher genera as follows: if G is a 2-cell embedding of an n-vertex graph with m edges in a surface of Euler genus g that has f faces, then $n+f \leq m+2-2g$. It follows that an n-vertex graph G, $n \geq 3$, that can be embedded in a surface of Euler genus g has at most $3n+6g-6$ edges.

Theorem (Heawood's Formula)**.** *Let G be a graph that can be embedded in a surface of Euler genus g. It holds that*

$$\chi(G) \leq \left\lfloor \frac{7+\sqrt{1+48g}}{2} \right\rfloor .$$

The bound given by Heawood's Formula is tight for every surface as the complete graph with the corresponding number of vertices can be embedded in the surface.

Exercises

1. Show that the list chromatic number of any even cycle is equal to two.

2. Show that the list chromatic number of $\Theta_{2,2,2m}$ is equal to two for every $m \in \mathbb{N}$.

3. Show that the list chromatic number of $\Theta_{4,4,4}$ is equal to three.

4. Show that the list chromatic number of a graph obtained from two copies of C_4 by joining them by an edge is equal to three.

The graph $\Theta_{a,b,c}$ is obtained by identifying the end-vertices of an a-edge path, a b-edge path and a c-edge path, i.e., the graph has $a+b+c-1$ vertices and $a+b+c$ edges.