Topics in Advanced Combinatorics
Lectures 3 and 4 (summary)

The aim of the two lectures is to show the following theorem.

Theorem. *The list chromatic number of every planar bipartite graph is at most three.*

The theorem follows from the following statement. We say that a graph G is L-colorable for a list assignment $L : V(G) \to 2^{\mathbb{N}}$ there exists $c : V(G) \to \mathbb{N}$ such that $c(v) \in L(v)$ for every $v \in V(G)$ and $c(u) \neq c(v)$ for every $uv \in E(G)$.

Theorem (Alon-Tarsi Theorem)**.** *Let G be a graph that has an orientation $\vec{G}$ such that the number of Eulerian subgraphs with an even number of edges and with an odd number of edges differ. The graph G is L-colorable for every list assignment L such that $|L(v)|$ is equal to the in-degree of v increased by one.*

To obtain the upper bound on the list chromatic number of planar bipartite graphs, we need the following theorem and lemmas.

Theorem (Hall's Theorem)**.** *Let $A_1, \ldots, A_k$ be sets such that*

$$\left| \bigcup_{j \in J} A_j \right| \geq |J|$$

for every $J \subseteq \{1, \ldots, k\}$. Then, there exist distinct $a_1, \ldots, a_k$ such that $a_i \in A_i$ for every $i \in \{1, \ldots, k\}$.

Lemma. *Every triangle-free n-vertex planar graph, $n \geq 3$, has at most $2n - 4$ edges.*

Lemma. *Every triangle-free planar graph has an orientation with the maximum in-degree at most two.*

To prove Alon-Tarsi Theorem, we need the following statement.

Theorem (Combinatorial Nullstellensatz)**.** *Consider an arbitrary field $\mathbb{F}$. Let $f(x_1, \ldots, x_n)$ be a polynomial such that the degree of x_i in f is t_i, $i \in \{1, \ldots, n\}$, and the coefficient of $x_1^{t_1} \cdots x_n^{t_n}$ is non-zero. If $S_1, \ldots, S_n$ are subsets of $\mathbb{F}$ such that $|S_i| \geq t_i + 1$ for every $i \in \{1, \ldots, n\}$, then there exist $s_1 \in S_1, \ldots, s_n \in S_n$ such that $f(s_1, \ldots, s_n) \neq 0$.*

Seminar material (for possible student presentations)

The following result was proven by Fleischner and Stiebitz [Discrete Mathematics 101 (1992), 39–48].

Theorem. *Let G be a 4-regular $3n$-vertex graph that has an edge-decomposition to a Hamilton cycle and n triangles. The list chromatic number of G is equal to three.*

A graph G is k-edge-colorable if there exists $c: E(G) \to \{1, \ldots, k\}$ such that $c(e) \neq c(e')$ for any two distinct edges e and e' sharing a vertex. A graph G is k-edge-list-colorable if for every $L: E(G) \to 2^{\mathbb{N}}$ such that $|L(e)| \geq k$ for every $e \in E(G)$, there exists $c: E(G) \to \{1, \ldots, k\}$ such that $c(e) \in L(e)$ for every $e \in E(G)$ and $c(e) \neq c(e')$ for any two distinct edges e and e' sharing a vertex.

Four Color Theorem is known to be equivalent to the statement that every planar cubic bridgeless graph is 3-edge-colorable. Ellingham and Goddyn [Combinatorica 16 (1996), 343–352] proved the following stronger result (assuming Four Color Theorem).

Theorem. *Every planar cubic bridgeless graph is* 3*-edge-list-colorable.*

Exercises

1. Every plane quadrangulation is 2-colorable.
2. Let G be an n-vertex planar graph. If G has no cycle of length smaller ℓ and $n \geq \frac{\ell+2}{2}$, then G has at most $\frac{\ell}{\ell-2}(n-2)$ edges.
3. Every k-regular bipartite graph is k-edge-colorable.