Topics in Advanced Combinatorics
Lecture 5 (summary)

In the previous lecture, we have seen that every plane quadrangulation is 2-colorable. The chromatic number of a quadrangulation of the torus can be two, three or four, however, the following remarkable theorem of Youngs excludes the middle option for the projective plane. Recall that the projective plane is the surface obtained from the disc by identifying antipodal points on the boundary, or alternatively, the projective plane can be obtained from the plane by adding a crosscap (the surface obtained by adding two crosscaps is the Klein bottle).

Theorem (Youngs' Theorem)**.** *The chromatic number of a quadrangulation of the projective plane is either two or four.*

In general, graphs with large minimum degree (and so many edges) can have small chromatic number, in particular, complete bipartite graphs $K_{n,n}$ are 2-chromatic. However, this is not the case for list chromatic number as proven by Alon.

Theorem. *For every $k \in \mathbb{N}$, there exists $d > 0$ such that every n-vertex graph with at least dn edges has list chromatic number at least k.*

We remark that Alon showed a stronger quantitative statement that the list chromatic number of a graph with average degree d is at at least $(0.5 - o(1)) \log_2 d$.