Topics in Advanced Combinatorics
Lectures 6 (summary)

In this lecture, we will be interested in the concept of nowhere-zero flows in graphs, which is a concept dual to that of graph coloring. A *digraph* throughout this lecture is a directed graph that may contain loops and parallel edges (regardless of their orientation). If A is an abelian group and G is a digraph, a function $f : E(G) \to A$ is an A-flow if the following holds for every vertex v of G:

$$\sum_{u \in N_G^-(v)} f(uv) = \sum_{u \in N_G^+(v)} f(vu).$$

An A-flow f is nowhere-zero if $f(e) \neq 0$ for every edge $e \in E(G)$.

We have established the following theorem (we define the notion of a k-flow later); also observe that if a graph G has orientation that admits a nowhere-zero A-flow, then every orientation admits a nowhere-zero A-flow.

Theorem. *The following statements are equivalent.*

- *Every planar graph is* 4*-colorable.*
- *Every planar* 3*-regular bridgeless graph is* 3*-edge-colorable.*
- *Every planar* 3*-regular bridgeless graph has an orientation that admits a nowhere-zero* $\mathbb{Z}_4$*-flow.*
- *Every planar* 3*-regular bridgeless graph has an orientation that admits a nowhere-zero* $\mathbb{Z}_2 \times \mathbb{Z}_2$*-flow.*
- *Every planar* 3*-regular bridgeless graph has an orientation that admits a nowhere-zero* 4*-flow.*

The equivalences are implied by the following lemmas.

Lemma. *A plane graph G is* 4*-colorable if and only if its dual G^* has an orientation that admits a nowhere-zero* $\mathbb{Z}_2 \times \mathbb{Z}_2$*-flow.*

Lemma. *A* 3*-regular bridgeless graph G is* 3*-edge-colorable if and only if G has an orientation that admits a nowhere-zero* $\mathbb{Z}_2 \times \mathbb{Z}_2$*-flow.*

Before the next lemma, we need to introduce some notation. If G is a digraph and A is an abelian group, then $\mathrm{NZF}_A(G)$ is the number of nowhere-zero A-flows of G.

Lemma. *Let G be a digraph and let A and A' be two abelian groups with the same number of elements. It holds that $NZF_A(G) = NZF_{A'}(G)$.*

We next introduce the notion of a k-flow. If G is a digraph, we say that a function $f : E(G) \to \mathbb{Z}$ is a k-flow if $f(e) \in \{1, \ldots, k-1\}$ for every edge e of G and the following holds for every vertex v of G:

$$\sum_{u \in N_G^-(v)} f(uv) = \sum_{u \in N_G^+(v)} f(vu).$$

A k-flow f is nowhere-zero if $f(e) \neq 0$ for every edge e.

Lemma. *A digraph G has a nowhere-zero k-flow if and only if G has a nowhere-zero $\mathbb{Z}_k$-flow.*

Exercises

1. Let G be a graph that contains two edge-disjoint spanning trees. Show that G has a nowhere-zero $\mathbb{Z}_4$-flow.

2. Show that for every digraph G, there exists a polynomial $p_G(x)$ such that $\mathrm{NZF}_A(G) = p_G(|A|)$.

3. Show that a plane graph G is 3-colorable if and only if there exists an orientation of its dual that admits a nowhere-zero $\mathbb{Z}_3$-flow.