Topics in Advanced Combinatorics
Lectures 7 (summary)

In this lecture, we present results of independent interest on our way towards proving that every bridgeless graph has an orientation admitting a nowhere-zero 8-flow. This is not the best possible: Seymour's Theorem asserts that every bridgeless graph has an orientation admitting a nowhere-zero 6-flow, and Tutte's Five Flow Conjecture postulates that every bridgeless graph has an orientation admitting a nowhere-zero 5-flow; this would be the best possible as the Petersen graph has no orientation admitting a nowhere-zero 4-flow. The condition of being bridgeless is necessary in these statements.

We will be concerned with perfect matchings in graphs. A *perfect matching* in a graph G is a subset $F \subseteq E(G)$ such that each vertex of G is incident with exactly one edge of F. Recall that a *bridge* is an edge whose removal from a graph increases the number of components and a graph G is *cubic* if each vertex of G has degree three, i.e., G is 3-regular.

Theorem (Petersen's Theorem)**.** *Every cubic bridgeless graph has a perfect matching.*

We proved using induction the following stronger statement.

Theorem. *Let G be a cubic bridgeless graph and e an edge of G. The graph G has a perfect matching F such that $e \notin F$.*

To prove the stronger statement, we utilized the following lemma.

Lemma. *Let G be a bridgeless cubic graph and let xy be any edge of G. Further, let x_1 and x_2 be the two neighbors of x different from y and let y_1 and y_2 be the two neighbors of y different from x. Let G_1 be the graph obtained from G by removing the vertices x and y and adding the edges x_1y_1 and x_2y_2, and let G_2 be the graph obtained from G by removing the vertices x and y and adding the edges x_1y_2 and x_2y_1. At least one of the graphs G_1 and G_2 is bridgeless.*

Using the observation that if F is a perfect matching in a graph G and a set $W \subseteq V(G)$ has an odd size, then the number of edges of F with exactly one end vertex in W is odd, we can derive the following corollary of Petersen's Theorem.

Corollary. *Let G be a cubic bridgeless graph and e an edge of G. The graph G has a perfect matching F such that $e \in F$.*

Exercises

1. Show that the Petersen graph is not 3-edge-colorable.

2. Show that for every G that has a Hamilton cycle, i.e., a spanning cycle, has an orientation that admits a nowhere-zero $\mathbb{Z}_2^2$-flow.

3. Let G be a bridgeless graph and let v be a vertex of degree at least four in G. Further, let $v_1, \ldots, v_4$ be any four neighbors of v and let G_i, $i \in \{1, 2, 3\}$, be the graph obtained from G by removing the edges vv_i and vv_{i+1} and adding the edge v_iv_{i+1}. Show that at least one of the graphs G_1, G_2 and G_3 is bridgeless.