Topics in Advanced Combinatorics
Lectures 8 (summary)

We started the lecture by a student presentation of the following result.

Theorem (Nash-Williams Theorem). *Let G be a multigraph and $k \in \mathbb{N}$. If for any partition of the vertex set $V(G)$ to sets $V_1, \ldots, V_\ell$ the number of edges with end vertices in different sets of the partition (crossing edges) is at least $k(\ell - 1)$, then G has k edge-disjoint spanning trees.*

For a (multi)graph G, a k-edge-cut is a partition of $V(G)$ to two parts such that the number of the edges between the two parts is exactly k. We obtain the following corollary of the Nash-Williams Theorem.

Corollary. *Let G be a multigraph that has no k-edge-cut for $k \in \{0, 1, 2, 3\}$. Then G has two edge-disjoint spanning trees.*

Recall that any graph G with two edge-disjoint spanning trees, in particular, any graph G satisfying the assumptions of the corollary, has an orientation that admits nowhere-zero $\mathbb{Z}_2^2$-flow.

To prove our main theorem, we need the following strengthening of results presented in the previous lecture.

Theorem. *Every cubic bridgeless graph G has a perfect matching F such that every 3-edge-cut of G has exactly one crossing edge that belongs to F.*

Our main theorem reads as follows.

Theorem. *Every bridgeless graph has an orientation that admits nowhere-zero $\mathbb{Z}_2^3$-flow.*